Optimization Methods in Management Science

MIT 15.053, Spring 2013

Problem Set 5, Due: Thursday April 2th, 2013

Problem Set Rules:

1. Each student should hand in an individual problem set.
2. Discussing problem sets with other students is permitted. Copying from another person or solution set is *not* permitted.
3. Late assignments will *not* be accepted. No exceptions.

Problem 1

(15 points) A local radio station is going to schedule commercials within 60 second blocks. Consider the following six commercials. number 1 is 12 seconds long, number 2 is 18 seconds long, number 3 is 22 seconds long, number 4 is 35 seconds long, number 5 is 40 seconds long and number 6 is 59 seconds long. What is the smallest number of 60 second blocks that the commercials fit into?

Solution. The key is to start with enough blocks for a trivial arrangement, in this case 5 obviously will work. Then we let $y_j = 1$ if block j is used and $y_j = 0$ otherwise and $x_{ij} = 1$ if commercial i is assigned to block j and $x_{ij} = 0$ otherwise. Based on this choice of decision variables, the formulation is as follows:

$$\begin{array}{lll} \min & y_1 + y_2 + y_3 + y_4 + y_5 & \\ \text{s.t.} & 12x_{1j} + 18x_{2j} + 22x_{3j} + 35x_{4j} + 40x_{5j} + 59x_{6j} \leq 60y_j, & \text{for } j = 1, \ldots, 5 \\ & x_{i1} + x_{i2} + x_{i3} + x_{i4} + x_{i5} = 1, & \text{for } i = 1, \ldots, 6 \\ & x_{ij}, y_j \in \{0, 1\} & \text{for } j = 1, \ldots, 5 \ \&\ i = 1, \ldots, 6 \end{array}$$

The first set of constraints represent the condition that each possible programming block contains at most 60 seconds of commercial programming. The second set of constraints represent the condition that each commercial is assigned to exactly one programming block. We then minimize the total number of blocks. Additionally, we need all of our decision variables to be binary.

By inspection, however, the minimum number of blocks needed is 4; the sum of all commercial times is 186, meaning that more than 3 are needed, and there is a clear way to use only 4.

Problem 2

(25 points) A typical large oil and gas company operates many explorations and production projects, which involve several billion dollars every year. These companies are annually faced with the problem of where the capital should be spent and which combination of projects should

be selected from several possible project mixes. They have the difficult task of portfolio selection from a large number of competing projects for immediate or future operation under a limited amount of investments.

Consider a small firm with 4 competing oil production projects. Table 1 presents the production, capital, and the net present value for these projects. They contact you to help select the best combination of projects under a certain amount of investment, while fulfilling the firm's goals.

(a) (5 points) Formulate an integer program to maximize the net present value (NPV) subject to a capital limit stating the firm can spend no more than $32 million (M$) and a production level stating the firm must produce at least 73 million barrels (Mbbl).

Solution. We define four binary variables x_A, x_B, x_C, x_D as follows:

$$x_A := \begin{cases} 1 & \text{if Project A is selected,} \\ 0 & \text{otherwise.} \end{cases} \qquad x_B := \begin{cases} 1 & \text{if Project B is selected,} \\ 0 & \text{otherwise.} \end{cases}$$

$$x_C := \begin{cases} 1 & \text{if Project C is selected,} \\ 0 & \text{otherwise.} \end{cases} \qquad x_D := \begin{cases} 1 & \text{if Project D is selected,} \\ 0 & \text{otherwise.} \end{cases}$$

Then our formulation is:

$$\begin{aligned} \max \quad & 25x_A + 20x_B + 19x_C + 28x_D \\ \text{s.t.} \quad & 11x_A + 9x_B + 14x_C + 17x_D \leq 32, \\ & 28x_A + 20x_B + 25x_C + 30x_D \geq 73, \\ & x_A, x_B, x_C, x_D \in \{0, 1\}. \end{aligned}$$

(b) (9 points, 3 points each) Suppose that there are the following additional constraints:

(i) If Project A is selected, then Project B is also selected;
(ii) Either Project A is selected or Project C selected, but not both;
(iii) At least one of Projects A,B, and D is selected.

Extend your integer program to satisfy theses constraints.

Solution. The additional constraints for each part are as follows:

(i) $x_A \leq x_B$ or equivalently $x_A - x_B \leq 0$;
(ii) $x_A + x_B = 1$.
(iii) $x_A + x_B + x_C \geq 1$.

(c) (5 points) Now assume that the selection must satisfy the production level as well as the budget limit over each of the next 3 years as indicated in Table 2. The production and capital for each project during the next 3 years are given in Table 3. Write an integer program to determine the most profitable selection.

Solution. Suppose that x_A, x_B, x_C, x_D are the binary variables as defined in Part A. Here we have a budget constraint and a production constraint for each year. Therefore,

Table 1: Portfolio optimization problem Data

Project	NPV (M$)	Capital (M$)	Production (Mbbl)
A	25	11	28
B	20	9	20
C	19	14	25
D	28	17	30

Table 2: Budget limitation and production level over the next 3 years

Year	Budget (M$)	Production (Mbbl)
1	18	20
2	10	25
3	7	30

the formulation is:

$$\begin{aligned}
\max \quad & 25x_A + 20x_B + 19x_C + 28x_D \\
\text{s.t.} \quad & 5x_A + 4x_B + 6x_C + 8x_D \leq 18, \\
& 4x_A + 3x_B + 5x_C + 5x_D \leq 10, \\
& 3x_A + 3x_B + 4x_C + 5x_D \leq 7, \\
& 5x_A + 5x_B + 6x_C + 8x_D \geq 20, \\
& 5x_A + 7x_B + 9x_C + 10x_D \geq 25, \\
& 8x_A + 8x_B + 10x_C + 12x_D \geq 30, \\
& x_A, x_B, x_C, x_D \in \{0, 1\}.
\end{aligned}$$

(d) (6 points) Generalize your integer program to a general setting: Assume that you are given a set of n projects and your task is to maximize the net present value (NPV), subject to a capital limit stating that we can spend no more that C_i and a production limit stating that we must produce at least P_i million barrels over each of the next L year. Assume that npv_j represents the NPV of jth asset and $p_{i,j}$ and $c_{i,j}$ represent the production and capital for the j^{th} project in the i^{th} year, respectively. Write the corresponding integer program.

Solution. We define the decision variables $x_1, x_2, \ldots, x_n$ as follows:

$$x_j := \begin{cases} 1 & \text{if Project j is selected,} \\ 0 & \text{otherwise.} \end{cases}$$

Then the problem is formulated as the following integer program:

$$\begin{aligned}
\max \quad & \sum_{j=1}^{n} npv_j x_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} c_{ij} x_j \le C_i, && \text{for } i = 1, \dots, L, \\
& \sum_{j=1}^{n} p_{ij} x_j \ge P_i, && \text{for } i = 1, \dots, L, \\
& x_j \in \{0, 1\}, && \text{for } j = 1, \dots, n.
\end{aligned}$$

Table 3: Portfolio optimization problem data over the next 3 years

Project	first year	second year	third year
Capital (M$)			
A	5	4	3
B	4	3	3
C	6	5	4
D	8	5	5
Production (Mbbl)			
A	5	5	8
B	5	7	8
C	6	9	10
D	8	10	12

Do not be scared by the fact that we have parameters $n, C_i, P_i, c_{ij}, p_{ij}$ instead of numbers! You can treat them just as you would treat numbers. Start by defining the decision variables.

Problem 3

(24 points) We have two binary variables $x_1, x_2 \in \{0, 1\}$. We want to represent the outcome of the three logical operations `AND`, `OR`, `XOR` applied on x_1 and x_2. The definition of these three operations is as follows:

- Let $w = (x_1 \texttt{ AND } x_2)$. w is 1 if and only if both x_1 and x_2 are 1, and is 0 otherwise.
- Let $y = (x_1 \texttt{ OR } x_2)$. y is 1 if and only if at least one of the variables x_1 and x_2 is 1, and is 0 otherwise.
- Let $z = (x_1 \texttt{ XOR } x_2)$. z is 1 if and only if the variables x_1 and x_2 have different values, and is 0 otherwise.

In Table 4, we give the value of each variable w, y, z in terms of the value of x_1 and x_2. The definition in the table is equivalent to the one given above.

Your goal in this problem is to define w, y, z using linear constraints only. For the `AND` and `OR` operations, you are not allowed to introduce additional variables.

	$x_2 = 0$	$x_2 = 1$	$x_2 = 0$	$x_2 = 1$	$x_2 = 0$	$x_2 = 1$
$x_1 = 0$	0	0	0	1	0	1
$x_1 = 1$	0	1	1	1	1	0
	w (AND)		y (OR)		z (XOR)	

Table 4: Value of w, y, z as a function of x_1, x_2.

(a) (8 points) Write a set of linear constraints that define $w = (x_1 \texttt{ AND } x_2)$. The constraints should only involve the variables w, x_1, x_2.

Solution. There are multiple acceptable formulations; one is the following:

$$\begin{aligned} &w \le x_1 \\ &w \le x_2 \\ &w \ge x_1 + x_2 - 1 \end{aligned}$$

(b) (8 points) Write a set of linear constraints that define $y = (x_1 \texttt{ OR } x_2)$. The constraints should only involve the variables y, x_1, x_2.

Solution. Again, there are multiple acceptable formulations; one is the following:

$$\begin{aligned} &y \ge x_1 \\ &y \ge x_2 \\ &y \le x_1 + x_2 \end{aligned}$$

(c) (8 points) Write a set of linear constraints that define $z = (x_1 \texttt{ XOR } x_2)$. You are allowed to introduce an auxiliary variable in this case. Thus, the constraints should involve the variables z, x_1, x_2, and possibly an additional variable a.

Solution. Again, there are multiple acceptable formulations; one is the following:

$$\begin{aligned} &z = x_1 + x_2 - 2a \\ &a \le x_1 \\ &a \le x_2 \\ &a \ge x_1 + x_2 - 1 \\ &a \in \{0, 1\} \end{aligned}$$

Problem 4

(36 points) We are given an integer program defined as follows:

$$\left.\begin{array}{rl} \max & 10x_1 + 22x_2 + 5x_3 + 15x_4 + 17x_5 + 12x_6 + 4x_7 \\ \text{s.t.:} & 5x_1 + 3x_2 + 8x_3 + 9x_4 + 16x_5 + 5x_6 + 10x_7 \le 700 \\ \forall i = 1,2,3 & x_i \in \{0,1\} \\ \forall i = 4,5,6,7 & 0 \le x_i \le 200. \end{array}\right\}$$

For each of the parts below, you are to add constraint(s) and possibly variables to ensure that the logical condition is satisfied by the integer program. Each part is independent; that is, no part depends on the parts preceding it. You do not need to repeat the integer programming objective or constraints given above. You may use the big M method for formulating constraint when it is appropriate. If you need to use the big M method, choose the answer that corresponds to the best possible value for the big M coefficient that appears in the formulation. (By "best possible value" we mean the smallest possible value such that the logical constraints modeled through the big M remain valid.)

(a) (3 points) Write a single linear constraint that is equivalent to the statement "If $x_2 = 1$ is selected, then $x_1 = 0$"

Solution. $x_1 + x_2 \leq 1$

(b) (3 points) Write a single linear constraint that is equivalent to the statement "x_1 and x_3 cannot both be 1".

Solution. $x_1 + x_3 \leq 1$

(c) (4 points) Add a single integer variable w_4 and a constraint that ensures that x_8 is divisible by 3 but not divisible by 6. (The remainder when dividing by 6 must be 3).

Solution. $x_8 - 6w_4 = 3$.

(d) (4 points) Add three binary variables w_5, w_6, and w_7 and two constraints that ensures that $x_5 = 9$ or 15 or 20.

Solution. $x_5 = 9w_5 + 15w_6 + 20w_7$ and $w_5 + w_6 + w_7 = 1$.

(e) (7 points) Add 3 binary variables w_1, w_2, and w_3 and at most 4 constraints so as to ensure that at least two of the following constraints is satisfied: (i) $x_4 \geq 50$, (ii) $x_5 \leq 25$, (iii) $x_6 + x_7 \leq 100$.

Solution. $x_4 \geq 50 - M(1 - w_1)$, $x_5 \leq 25 + M(1 - w_2)$, $x_6 + x_7 \leq 100 + M(1 - w_3)$ and $w_1 + w_2 + w_3 \geq 2$, with $M \geq 300$.

(f) (7 points) Add variable(s) and constraint(s) that ensure either $2x_4+x_5 \leq 50$ or $4x_4-x_5 \geq 20$, but not both.

Solution. $2x_4 + x_5 \leq 50 + M(1 - w_8)$, $4x_4 - x_5 \geq 20 - M(1 - w_9)$, $w_8 + w_9 = 1$, with $M \geq 450$.

(g) (8 points) Add variable(s) and constraint(s) that model the cost of x_4 as $f_4(x_4)$, which is defined as follows: If $0 \leq x_4 \leq 50$, then $f_4(x_4) = 20x_4$. If $51 \leq x_4 \leq 100$, then $f_4(x_4) = 1000$. If $101 \leq x_4 \leq 200$, then $f_4(x_4) = -500 + 15x_4$.

Solution.

$$
\left.
\begin{array}{rrcl}
\text{New Obj Fn: max} & 10x_1 + 22x_2 + 5x_3 + f_4(x) + 17x_5 + 12x_6 + 4x_7 & & \\
\text{s.t.:} & w_1 + w_2 + w_3 & = & 1 \\
 & y_1 + y_2 + y_3 & = & x_4 \\
 & 20y_1 + 1000w_2 - 500w_3 + 15y_3 & = & f_4(x_4) \\
 & 0w_1 \ \leq \ y_1 & \leq & 50w_1 \\
 & 51w_2 \ \leq \ y_2 & \leq & 100w_2 \\
 & 101w_3 \ \leq \ y_3 & \leq & 200w_3 \\
 & w_1, w_2, w_3 & \in & \{0, 1\} \\
 & 0 \leq y_1, y_2, y_3 \leq 200. & &
\end{array}
\right\}
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