

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Department of Physics

Physics 8.901: Astrophysics I

Spring Term 2006

PROBLEM SET 5

Due: Thursday, April 6 in class

Reading: Finish reading the first part of Chapter 2 (§2.1–2.6) on stellar evolution in Hansen, Kawaler, & Trimble, and also read §9.3 on the solar neutrino problem. You may also find it useful to review §3.5–3.6 on degenerate equations of state. For a deeper discussion, you can consult Chapters 2–3 of *Black Holes, White Dwarfs, and Neutron Stars* by Shapiro & Teukolsky.

1. Lower and upper main sequence stars.

- (a) Estimate the hydrogen-burning lifetime of stars on the lower and upper ends of the main sequence. The lower end of the main sequence occurs near $0.085 M_{\odot}$, with $\log T_e = 3.438$ and $\log(L/L_{\odot}) = -3.297$. Assume that the upper end of the main sequence occurs at $90 M_{\odot}$ with $\log T_e = 4.722$ and $\log(L/L_{\odot}) = 6.045$. Assume that the $0.085 M_{\odot}$ star is entirely convective so that, through convective mixing, all of its hydrogen becomes available for burning rather than just the inner 10%.
- (b) Use the information above to calculate the radius of a $0.085 M_{\odot}$ star and a $90 M_{\odot}$ star on the main sequence.

2. **Evolution on the red giant branch.** Once a low-mass star leaves the main sequence and ascends the red giant branch, its luminosity is provided by hydrogen burning in a thin shell surrounding a degenerate helium core. As hydrogen is burned in the shell, the mass of the helium core grows and the star continues to ascend the giant branch. It turns out that both the luminosity and the radius of red giants can be written as steep functions of their helium core mass alone, nearly independent of any other quantity. These expressions are

$$L \simeq 2 \times 10^5 L_{\odot} \left(\frac{m_c}{M_{\odot}} \right)^6, \quad R \simeq 3700 R_{\odot} \left(\frac{m_c}{M_{\odot}} \right)^4,$$

where m_c is the helium core mass, and L and R are the luminosity and radius of the (entire) red giant, respectively. Note that these expressions are “empirical” in that they are based on the results of stellar evolution calculations and are *not* derived analytically.

Use these relations to compute the evolution of a star (L and R as functions of time) as it ascends the red giant branch. Take the initial core mass value to be $m_c = 0.1 M_{\odot}$ and the final value (at the tip of the giant branch) to be $m_c = 0.45 M_{\odot}$. (Note that $L = 0.007 \dot{M} c^2$, where \dot{M} represents both the rate that hydrogen is burned in the shell around the core, and the rate of growth of the degenerate helium core.)

Plot the radius, luminosity, and effective temperature of the star as a function of time on the giant branch. Also, plot the track of the evolving giant on an H-R diagram (i.e., $\log L$ versus $\log T$, with $\log T$ increasing to the left).

3. **Core helium flash in red giants.** (Adapted from Hansen, Kawaler, & Trimble, Problem 6.8).

Suppose you have a gram of pure ${}^4\text{He}$ in the center of a pre-helium-flash red giant. The density and temperature of the gram are, respectively, $\rho = 2 \times 10^5 \text{ g cm}^{-3}$ and $T = 1.5 \times 10^8 \text{ K}$. This is hot enough to burn helium by the triple- α reaction, which is the only reaction you will use. The energy generation rate for this reaction is given by

$$\epsilon_{3\alpha} = \frac{5.1 \times 10^8 \rho^2 Y^3}{T_9^3} e^{-4.4027/T_9} \text{ erg g}^{-1} \text{ s}^{-1},$$

where Y is the helium mass fraction and T_9 is the temperature in units of 10^9 K .

You are now to follow the time evolution of the gram as helium burning proceeds, by computing the temperature $T(t)$ as a function of time. Start the clock running at $t = 0$ with the stated conditions. For simplicity, assume that the density and the helium concentration remain constant for all time and that no heat is allowed to leave the gram. Compute $T(t)$ numerically until that time when the material begins to become nondegenerate. For this, use the nonrelativistic demarcation line $\rho \approx 10^{-8} T^{3/2}$; densities below this will be nondegenerate.

For a given input of heat energy ΔE , you can calculate the change in temperature of a mass Δm from the specific heat c_V using

$$\Delta T = \frac{(\Delta E/\Delta m)}{c_V}.$$

Note that you will need to use the total specific heat $c_V = c_{V,e} + c_{V,\text{He}}$. The degenerate electron specific heat is given by

$$c_{V,e} = \frac{1.35 \times 10^5}{\rho} T x (1 + x^2)^{1/2} \text{ erg g}^{-1} \text{ K}^{-1},$$

where $\rho \approx 2 \times 10^6 x^3$ allows solution for the dimensionless Fermi momentum x . The ideal gas specific heat for the helium ions is

$$c_{V,\text{He}} = \frac{3k}{4m_p}.$$

Plot temperature $T(t)$ versus time in days. You will be able to recognize the flash when it happens because the temperature will suddenly skyrocket after not too many days of burning. Indicate whether the result you get is reasonable and what criteria you use to make that determination.

After how many days does the helium flash occur? As a test of the quality of your numerical integration scheme, do your best to determine this onset time precisely; to do so, you will need to either use a small step-size in time or an adaptive step-size method (see, e.g., Chapter 16 of *Numerical Recipes* by Press et al. or another reference on numerical integration of ordinary differential equations).

4. **Neutrino oscillations.** A promising solution to the solar neutrino problem (the deficit of electron neutrinos detected from the Sun relative to what is predicted by standard solar models; see your reading in §9.3 of Hansen, Kawaler, & Trimble) the phenomenon of neutrino oscillations, an expected consequence of non-zero neutrino masses. We can use the elementary quantum mechanics to understand the essential idea. For simplicity, we will ignore the τ -neutrino and consider only a two-state system comprising the electron neutrino ν_e and the muon neutrino ν_μ .

Our two-state system can be represented with two different sets of basis vectors. One set comprises the energy eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$; a neutrino of type j with momentum p is an energy eigenstate with

eigenvalue $E = \sqrt{p^2 c^2 + m_j^2 c^4}$. The other set comprises the weak-interaction eigenstates produced in weak decay processes, denoted by $|\nu_e\rangle$ and $|\nu_\mu\rangle$. Neutrinos are produced by weak interactions in eigenstates of lepton number (ν_e and ν_μ) that are not necessarily energy eigenstates. These two basis sets are related by a unitary transformation,

$$\begin{aligned} |\nu_e\rangle &= \cos\theta |\nu_1\rangle + \sin\theta |\nu_2\rangle \\ |\nu_\mu\rangle &= -\sin\theta |\nu_1\rangle + \cos\theta |\nu_2\rangle, \end{aligned}$$

where the phases have been chosen the phases to make the matrix elements real. The mixing angle θ is presumably determined by the same dynamics that determines the neutrino masses, and it is just as mysterious at this time.

Suppose that at time $t = 0$ we have a beam of electron neutrinos $|\nu_e\rangle$ (which is thus a superposition of mass eigenstates) with momentum p , propagating from the origin in the z -direction. For simplicity, use free-particle plane waves and non-relativistic quantum mechanics in what follows (even though, strictly speaking, one should be using wave packets and relativistic quantum mechanics).

- (a) In the limit where the masses m_j are small, show that the wavefunction for the beam after a time t and distance z is

$$\psi(z, t) = \exp\left[\frac{-ipc(t - z/c)}{\hbar}\right] \left(\exp(-im_1^2 c^3 t / 2\hbar p) \cos\theta |\nu_1\rangle + \exp(-im_2^2 c^3 t / 2\hbar p) \sin\theta |\nu_2\rangle \right).$$

Taking $E \approx pc$ and $z \approx ct$ and ignoring the overall phase factor, this reduces to

$$\psi(z, t) = \exp(-im_1^2 c^3 z / 2\hbar E) \cos\theta |\nu_1\rangle + \exp(-im_2^2 c^3 z / 2\hbar E) \sin\theta |\nu_2\rangle.$$

- (b) Show that, after a distance z , the probability that a muon neutrino ν_μ will be found in the initially pure ν_e beam is

$$P(z) = \sin^2(2\theta) \sin^2\left(\frac{c^3 \Delta m^2 z}{4\hbar E}\right),$$

where $\Delta m^2 \equiv m_1^2 - m_2^2$. We thus see that the initially pure $|\nu_e\rangle$ state oscillates until P reaches a maximum possible value of $\sin^2(2\theta)$, and then oscillates back to being a pure $|\nu_e\rangle$ state as it travels an oscillation length

$$L_{\text{osc}} = \frac{4\pi\hbar E}{c^3 \Delta m^2}.$$

We also see that, for neutrino oscillations to occur, we must have a non-zero mixing angle θ and a non-zero neutrino mass difference, and therefore at least one non-zero neutrino mass.

In principle, one could imagine a choice of Δm^2 and $\sin^2(2\theta)$ that could reproduce the observed deficits of solar neutrinos at various energies, although this would imply that the experimental results depend upon the precise distance between the Earth and the Sun (and that the solar neutrino problem would appear different on, say, Mars). A more likely explanation additionally involves a mechanism for enhanced oscillations during passage of neutrinos through the solar interior, due to the so-called MSW effect.

As of summer 2004, the best experimental values give $\sin^2 2\theta = 0.82 \pm 0.07$ and $\Delta m^2 c^4 = (8.2 \pm 0.6) \times 10^{-5} \text{ eV}^2$.

5. **Mass-radius relation for polytropic stars (e.g., white dwarfs).** In this problem, we return to polytropic stars, whose pressure and density are related as $P = K\rho^\gamma$, where $\gamma = 1 + (1/n)$, K and n are constants, and n is the polytropic index. Recall that these stars satisfy the Lane-Emden equation. In lecture and in a previous problem set, we showed that the radius of the star is $R = a\xi_1$ where

$$a = \left[\frac{(n+1)K}{4\pi G} \right]^{1/2} \rho_c^{(1-n)/2n},$$

and that the mass is

$$M = 4\pi a^3 \rho_c \left(-\xi^2 \frac{d\theta}{d\xi} \right)_{\xi=\xi_1}.$$

- (a) Show that the mass-radius relation for a polytropic star is

$$M = 4\pi R^{(3-n)/(1-n)} \left[\frac{(n+1)K}{4\pi G} \right]^{n/(n-1)} \xi_1^{(n-3)/(1-n)} \left(-\xi^2 \frac{d\theta}{d\xi} \right)_{\xi=\xi_1}.$$

- (b) Show that for a low-density white dwarf (non-relativistic, $\gamma = 5/3$, $n = 3/2$), the radius and mass can be expressed as

$$\begin{aligned} R &= 1.122 \times 10^4 \left(\frac{\rho_c}{10^6 \text{ g cm}^3} \right)^{-1/6} \left(\frac{\mu_e}{2} \right)^{-5/6} \text{ km}, \\ M &= 0.4964 \left(\frac{\rho_c}{10^6 \text{ g cm}^3} \right)^{1/2} \left(\frac{\mu_e}{2} \right)^{-5/2} M_\odot \\ &= 0.7011 \left(\frac{R}{10^4 \text{ km}} \right)^{-3} \left(\frac{\mu_e}{2} \right)^{-5} M_\odot, \end{aligned}$$

where μ_e is the mean molecular weight per electron and $n_e = \rho/\mu_e m_p$ where m_p is the proton mass, ρ is the mass density, and n_e is the electron number density. Note that for a completely ionized element of atomic number Z and weight A , $\mu_e = A/Z$ to 1 part in 10^4 .

- (c) Show that for a high-density white dwarf (relativistic, $\gamma = 4/3$, $n = 3$), the radius and mass can be expressed as

$$\begin{aligned} R &= 3.347 \times 10^4 \left(\frac{\rho_c}{10^6 \text{ g cm}^3} \right)^{-1/3} \left(\frac{\mu_e}{2} \right)^{-2/3} \text{ km}, \\ M &= 1.457 \left(\frac{\mu_e}{2} \right)^{-2} M_\odot. \end{aligned}$$

This is an approximation to the Chandrasekhar mass, the maximum mass of a stable white dwarf. (Strictly speaking, a more rigorous calculation would include electrostatic and other corrections to the degenerate equation of state.)