8.821 String Theory
Fall 2008

8.821 F2008 Problem Set 1 Solutions

I. BRANES ENDING ON BRANES

Let's first remind ourselves why branes typically can't end. A D$(p-2)$ brane action typically contains a term

$$S \supset \int_{D(p-2)} C_{p-1} \tag{1}$$

where C_{p-1} is a $(p-1)$ form gauge potential whose gauge transformation is given by $\delta C_{(p-1)} = d\Lambda_{p-2}$, Λ_{p-2} being a $(p-2)$ form parametrizing the gauge freedom (compare with familiar electromagnetism, which has A_1 and Λ_0). Under this gauge transform the variation of the action is

$$\delta S = \int_{D(p-2)} d\Lambda_{p-2} = \int_{\partial D(p-2)} \Lambda_{p-2} \tag{2}$$

where we have used Gauss's law in fancy differential forms notation $\int_{\mathcal{M}} dF = \int_{\partial\mathcal{M}} F$. This gauge variation must be 0 for all Λ_{p-2}; this means that the boundary $\partial D(p-2)$ must not exist, and thus the brane has no end.

On the other hand, consider adding the term mentioned in the problem set $S_{\text{new}} = \int_{D_p} F \wedge C_{p-1}$, where D_p is some other p-brane. The gauge variation of this new term is

$$\delta S_{\text{new}} = \int_{D_p} F \wedge d\Lambda_{p-2} = -\int_{D_p} dF \wedge \Lambda_{p-2} \tag{3}$$

(where we have assumed that the Dp brane at least has no boundary to discard a boundary term) and thus the new condition for the gauge invariance of the action is

$$\delta S_{\text{total}} = \int_{\partial D(p-2)} \Lambda_{p-2} - \int_{D_p} dF \wedge \Lambda_{p-2} = 0 \tag{4}$$

This condition can be satisfied only if the boundary $\partial D(p-2)$ sits on the Dp brane and $dF \neq 0$ on $\partial D(p-2)$. In that case the $D(p-2)$ brane can certainly end on the Dp brane.

What does this look like from the point of view of the worldvolume theory on the Dp brane? Recall the equations of motion for a 2−form field strength F,

$$d \star F = \star j_e \qquad dF = \star j_m \tag{5}$$

in terms of electric and magnetic currents j_e and j_m. $dF \neq 0$ thus implies the existence of something coupling magnetically to the worldvolume gauge field. Some fun numerology shows that on a $(p+1)$ dimensional worldvoume, j_e is always a 1-form (the gauge field couples electrically to a point-particle sweeping out a one-dimensional trajectory) but j_m is a $(p-2)$ form; in our case this $(p-2)$ form is related to the volume form of the $(p-2)$ dimensional boundary $\partial D(p-2)$.

Finally lets specialize to $p = 3$; in that case we have $D1$ branes ending on $D3$ branes; their endpoints are just points that look like magnetic monopoles to the $d = 4$ Yang-Mills theory living on the D3 brane.

II. TIMELIKE OSCILLATORS ARE EVIL

We have some offensive oscillators with $[a, a^\dagger] = -1$. Let's do the problems in reverse order. Consider the normal vacuum defined by $a|0\rangle = 0$ and compute the norm of $|1\rangle \equiv a^\dagger|0\rangle$. We have

$$\langle 0|aa^\dagger|0\rangle = \langle 0| - 1 + a^\dagger a|0\rangle = -1\langle 0|0\rangle \tag{6}$$

Thus if the vacuum has positive norm the excited states have negative norm and vice-versa. Unfazed, we consider a "funky vacuum" $|0_f\rangle$ defined by $a^\dagger|0_f\rangle = 0$. Funky excited states defined by $|n_f\rangle = a^n|0_f\rangle/n!$ will have positive norm, and we appear to have a nice Hilbert space. We write the Hamiltonian as $H = (E_0 - 1) - aa^\dagger$ to suit our funky interpretation. Unfortunately if we compute the energy of $|1_f\rangle$ we get

$$H|1_f\rangle = (E_0 - 1 - aa^\dagger)a|0_f\rangle = (E_0 - 2)a|0_f\rangle \tag{7}$$

and the energy is less than that of the ground state $(E_0 - 1)$! The energy keeps getting more negative the more we excite the funky vacuum, and we see that it is unbounded below. We conclude that timelike oscillators are evil.

III. EXTREMAL REISSNER-NORDSTROM BLACK HOLE

We start with the action

$$S_{\rm EM} = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\left(R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \tag{8}$$

and vary with respect to $g^{\mu\nu}$. The variation of the Einstein-Hilbert term R gives the normal Einstein tensor (see e.g. Carroll's GR book chapter 4). Varying the Maxwell part has two pieces; one part from the variation of $\sqrt{-g}$, which (leaving out the overall $1/16\pi G_N$) is $+\frac{1}{8}F_{\rho\sigma}F^{\rho\sigma}g_{\mu\nu}$ and one from the variation of the metric tensors tying together the indices, which is $-\frac{1}{2}F_{\mu\rho}F^{\rho}_{\nu}$. The full equations of motion work out to be

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\frac{1}{8}F^2 g_{\mu\nu} + \frac{1}{2}F_{\mu\rho}F^{\rho}_{\nu} \tag{9}$$

which is of the form claimed in the problem set.

Before computing the curvature tensors, note that taking the trace gives $R = 0$ as the right-hand side is identically traceless. The fact that the stress tensor of electromagnetism is traceless is a consequence of the classical scale invariance of the Maxwell action and implies that a spacetime supported only by the Maxwell action has vanishing Ricci scalar.

We will eventually need the curvature tensors in a ten-dimensional spacetime; thus for generality we will find them for a D dimensional spacetime containing a d-dimensional flat Poincare piece parametrized by x^i and a n dimensional flat Euclidean piece (the "transverse space") parametrized by y^a. Our ansatz for the metric is

$$ds^2 = \frac{1}{H^2(y)}\eta_{ij}dx^i dx^j + H^2(y)\delta_{ab}dy^a dy^b \tag{10}$$

This is slightly more general than the ansatz in the problem set as that only had H depending on one of the y coordinates whereas we are letting it depend on all of them. Also note I am not using polar coordinates on this transverse space but flat ones; the conversion to polar is trivial at the end but for now the flat coordinates simplify life quite a bit. We now compute the curvature. This is straightforward but tedious [1]. I am not providing details of the calculation, as a much nicer explanation of a similar (slightly simpler) calculation can be found in Carroll, Appendix J. The answer is

$$R_{\mu\nu} = \frac{\eta_{\mu\nu}}{H^4}\left[\frac{(\partial H)^2}{H^2}(n-d-3) + \frac{\Box H}{H}\right] \tag{11}$$

$$R_{ab} = \frac{\partial_a H\partial_b H}{H^2}(2n-4-4d) + \frac{\partial_a\partial_b H}{H}(2-n+d) + \delta_{ab}\left(\frac{(\partial H)^2}{H^2}(d+3-n) - \frac{\Box H}{H}\right) \tag{12}$$

Here $(\partial H)^2 = \sum_a(\partial_a H)^2$ and $\Box H = \sum_a \partial_a^2 H$; these are all with respect to the "transverse space". Ok, for now we just set $n = 3$, $d = 1$. This gives

$$R_{tt} = -\frac{1}{H^4}\left[\frac{-(\partial H)^2}{H^2} + \frac{\Box H}{H}\right] \tag{13}$$

$$R_{ab} = -2\frac{\partial_a H\partial_b H}{H^2} + \delta_{ab}\left(\frac{(\partial H)^2}{H^2} - \frac{\Box H}{H}\right) \tag{14}$$

and taking the trace of this we find the Ricci scalar

$$R = -\frac{2}{H^3}\Box H = 0 \tag{15}$$

which we know is 0 by the Einstein equations. Thus it appears that we need H to be a harmonic function on the transverse space. Next we turn to the Maxwell equation of motion

$$\nabla_\mu F^{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}F^{\mu\nu}) = 0 \tag{16}$$

[1] By which I mean that it was incredibly un-straightforward for me, but in principle if you make no algebra mistakes it isn't too bad.

Plugging in the ansatz $F_{ta} = b\frac{\partial_a H}{H^2}$ we find that all factors of H^2 cancel and this becomes

$$\sum_a \partial_a\partial_a H = \Box H = 0 \tag{17}$$

Thus Maxwell equations are also satisfied. We are not done yet, we also need to verify that all of the components of Einstein's equations (9) are satisfied. We write the intermediate results

$$F^2 = -2b^2\frac{(\partial H)^2}{H^4} \qquad F_{t\rho}F_t^\rho = b^2\frac{(\partial H)^2}{H^6} \qquad F_{a\rho}F_b^\rho = -b^2\frac{\partial_a H\partial_b H}{H^2} \tag{18}$$

Combining these to form the right-hand side of (9) we see that all the Einstein equations can be satisfied provided $b = 2$. Hooray! We now write down the most general form of H (subject to the boundary condition that $H(y \to \infty) = 1$. We know what this is from old-fashioned electromagnetism; it is the potential from a series of point charges.

$$H(y^a) = 1 + \sum_i \frac{A_i}{|y - y_i|} \tag{19}$$

where $\{y_i\}$ provides a set of 3-vectors and A_i a set of corresponding "charges", both of which are arbitrary. This function is harmonic everywhere except at the poles $y = y_i$, and it is an *exact* solution to the Einstein-Maxwell equations. It describes a multi-center black hole solution, where we have a set of black holes with centers located at y_i, held somehow in a stable configuration by a delicate interplay of gravitational and electromagnetic forces. Beautiful.

What is A_i? To find this we finally focus on a one black-hole solution and use polar coordinates. We also let H depend only on ρ. We obtain $F_{\rho t} = -2\frac{\partial_\rho H}{H^2}$, and

$$Q \equiv \int_{S^2} \star F = -\int_{S^2} F_{\rho t}H^2\rho^2\sin(\theta)d\theta\wedge d\phi = \int d\theta d\phi\sin(\theta)2A = 8\pi A \tag{20}$$

Thus the full (one-black hole) metric is

$$ds^2 = -\frac{1}{H^2}dt^2 + H^2(d\rho^2 + \rho^2 d\Omega_2) \qquad H(\rho) = 1 + \frac{Q}{8\pi\rho} \tag{21}$$

Next, we look at the "near-horizon" limit, which is where $\rho \to 0$. We obtain

$$ds^2 = -\rho^2\left(\frac{8\pi}{Q}\right)^2 dt^2 + \left(\frac{Q}{8\pi\rho}\right)^2 d\rho^2 + \left(\frac{Q}{8\pi}\right)^2 d\Omega_2 \tag{22}$$

This is the metric on $AdS_2 \times S^2$ with AdS length $L = \frac{Q}{8\pi}$.

Finally, let us add some magnetic charge. To get the solution, let's use the fact that the Hodge dual of a solution with electric charge is one with magnetic charge. We take the dual of a solution with a pure radial electric field F_e

$$F_m = \star F_e = b\partial_\rho H\sin(\theta)\rho^2 d\theta\wedge d\phi \tag{23}$$

Now we superpose these two solutions to obtain something *dyonic*, i.e. with both electric and magnetic charge

$$F_{\text{dyonic}} = \partial_\rho H\left(\frac{b}{H^2}d\rho\wedge dt + a\rho^2\sin(\theta)d\theta\wedge d\phi\right) \tag{24}$$

From here the analysis proceeds exactly as above. I will not go into details; suffice to say that we now require $a^2 + b^2 = 4$ and the final solution with electric charge Q and magnetic charge P has metric given by (21) except now with

$$H(\rho)_{\text{dyonic}} = 1 + \frac{\sqrt{Q^2+P^2}}{8\pi\rho} \tag{25}$$

IV. RR SOLITON

Despite its horrifying name and birthplace, this is actually almost exactly the same problem as the one we just did, only with 2.5 times as many indices. We start with the action

$$S_{IIB} = \frac{1}{16\pi G_N} \int d^{10}x\sqrt{-g}\left(e^{-2\phi}(R + 4\partial_M\phi\partial^M\phi) - \frac{1}{5!}(F^5)^2\right) \tag{26}$$

We now need to vary this with respect to g^{MN}. Note that if the dilaton ϕ is not constant this is a bit trickier than you might expect, as the variation of the Einstein-Hilbert term will result in extra pieces that are normally a total derivative but now are not. Thus the Einstein equations of motion will contain extra pieces depending on gradients of the dilaton. Luckily we will examine only configurations where the dilaton is constant, so I will not keep track of these terms but you should know that they are there. Anyway, we obtain simply

$$e^{-2\phi}\left(R_{MN} - \frac{1}{2}g_{MN}R\right) = -\frac{1}{2.5!}(F^5)^2 + \frac{1}{4!}F_{M....}F_N^{....} \tag{27}$$

Taking the trace again we obtain as before $R = 0$ (The RR five-form is classically scale invariant in ten dimensions, etc. etc.) A new ingredient is provided by the self-duality of the five-form; this means that $F = \star F$, and thus $F^2 \sim F \wedge \star F = F \wedge F = 0$. The equation becomes

$$R_{MN} = \frac{e^{2\phi}}{4!}F_{M....}F_N^{....} \tag{28}$$

Recall also that the ansatz is

$$ds^2 = \frac{1}{\sqrt{h(y)}}\eta_{\mu\nu}dx^\mu dx^\nu + \sqrt{h(y)}\delta_{ab}dy^a dy^b \tag{29}$$

Luckily we already worked out the curvature, once we make the substitution $H^2 = \sqrt{h}$, $n = 6, d = 3$ in the formulas (11) and (12):

$$R_{\mu\nu} = \frac{\eta_{\mu\nu}}{4h}\left[\frac{\Box h}{h} - \frac{(\partial h)^2}{h^2}\right] \tag{30}$$

$$R_{ab} = -\frac{1}{2}\frac{\partial_a h\partial_b h}{h^2} + \frac{\delta_{ab}}{4}\left(\frac{(\partial h)^2}{h^2} - \frac{\Box h}{h}\right) \tag{31}$$

We now proceed in direct analogy to what we did above. Taking the trace we find $R = -\frac{1}{2}\frac{\Box h}{h} = 0$, implying that h is harmonic on the transverse space. We should check that the Maxwell equations for the five-form are satisfied. These equations are just

$$\partial_M(\sqrt{-g}F^{MNOPQ}) = 0 \tag{32}$$

Our ansatz for F is

$$F = b(1+\star)\, dt \wedge dx \wedge dy \wedge dz \wedge d(h^{-1}) \tag{33}$$

which satisfies $F_{txyza} = -\frac{b}{h^2}\partial_a h$. Plugging this into the equation above and using $g^{\mu\nu} = \sqrt{h}\eta^{\mu\nu}$, $g^{ab} = 1/\sqrt{h}\delta^{ab}$, $\sqrt{-g} = \sqrt{h}$, we get

$$\partial_a\left[\sqrt{h}(\sqrt{h})^4(\sqrt{h})^{-1}F_{txyza}\right] = -b\partial_a\left[h^2\frac{\partial_a h}{h^2}\right] = 0, \tag{34}$$

implying yet again that the Maxwell equation is satisfied iff h is harmonic. By now you probably believe me that this is a solution, but we should conscientiously check that all of the other components of the Einstein equations are satisfied. If we skipped this step, not only would we not have the pleasure of contracting F^5 with itself in entertaining ways, but we would also not be able to fix the value of b.

So we compute the right-hand side of the Einstein equation, $F_{M....}F_N^{....}$. The easy one first:

$$F_{\mu....}F_\nu^{....} = -4!b^2\frac{(\partial h)^2}{h^3}\eta_{\mu\nu} \tag{35}$$

To understand this, note that if we fix say the two indices t in $F_{t....}F_t^{....}$ and let all the rest vary, it is clear from the form (33) that to contribute the remaining indices must be a permutation of $\{x, y, z, y^a\}$, where all a can contribute (I apologize for the bad notation; y with no superscript is one of the coordinates along the Poincare piece x^μ. y^a with a superscript is a coordinate along the transvers space). There are 4! such permutations for each choice of y^a, resulting in the structure shown. Now the hard one, the computation of $F_{a....}F_b^{....}$. To do this we first note that

$$\star(dt \wedge dx \wedge dy \wedge dz \wedge dy^1) \equiv \frac{1}{5!}\epsilon^{txyzy^1}{}_{MNOPQ} dx^M \wedge dx^N \wedge dx^N \wedge dx^O \wedge dx^P \wedge dx^Q = \tag{36}$$

$$= h^2 dy^2 \wedge dy^3 \wedge dy^4 \wedge dy^5 \wedge dy^6 \tag{37}$$

Thus for example we see from (33) that $F_{23456} \equiv F_{y^2y^3y^4y^5y^6} = -b\partial_1 h$; if all the indices are y^a's, then the value of the component is the derivative of h with respect to the one y that isn't in the string. Now we are ready. There are two cases, start with $a \neq b$ and take $a = 1, b = 2$ for concreteness. Then

$$F_{1....}F_2^{....} = F_{1\mu\nu\rho\sigma}F_2^{\mu\nu\rho\sigma} + F_{1abcd}F_2^{abcd} = 4!\left(-h^2 F_{1txyz}F_{2xyzt} + \frac{1}{h^2}F_{13456}F_{23456}\right) \tag{38}$$

$$= -2.4!b^2\frac{\partial_1 h \partial_2 h}{h^2} \tag{39}$$

Thus for $a \neq b$, $F_{a....}F_b^{....} = -2.4!b^2\frac{\partial_a h \partial_b h}{h^2}$. Now the case $a = b = 1$; going through an analogous computation we get

$$F_{1....}F_1^{....} = F_{1\mu\nu\rho\sigma}F_1^{\mu\nu\rho\sigma} + F_{1abcd}F_1^{abcd} = \frac{4!b^2}{h^2}\left(-\partial_1 h \partial_1 h + \sum_{a\neq 1}\partial_a h \partial_a h\right) \tag{40}$$

We can summarize these two relations in the combined form

$$F_{a....}F_b^{....} = 2.\frac{4!b^2}{h^2}\left(-\partial_a h \partial_b h + \frac{\delta_{ab}}{2}(\partial h)^2\right) \tag{41}$$

Comparing with (30) and the equation of motion (27), we see that the Einstein equations are satisfied if the value of b is fixed to be $b = \pm e^{-\phi}/2$. Hooray! Once again we see that we have a beautiful multi-center solution

$$h(y) = 1 + \sum_i \frac{R_i^4}{|y - y^i|^4} \tag{42}$$

where we can move the centers y^i of the D3 branes wherever we like in the transverse 6-dimensional space and they will stay where we leave them due to the delicate yet strangely robust cancellation of the RR five-form field with gravity. As mentioned in lecture, this bizarre property manifests itself in the dual field theory description of this configuration–remember from lecture 6 that the X^i in $\mathcal{N} = 4$ SYM parametrize movement in a $6N$ dimensional space along which the potential of the field theory is flat; there is no energy cost associated with changes in X. Now you know "why".

Finally we restrict to one D3 brane in a spherically symmetric setup. Doing the integral $Q = \int_{S^5}\star F = 2e^{-\phi}R^4\pi^3$ we find that $R^4 \sim g_s Q$, where we have used $g_s = e^\phi$. Q is the RR charge, and is quantized in appropriate units. Essentially we have found that

$$R^4 \sim l_s^4 g_s N \tag{43}$$

where l_s is the string length scale and N is the integer number of RR flux quanta. This is the reason why AdS/CFT maps a classical gravity setup to a dual configuration with a large t'Hooft coupling–we need $g_s N = g_{YM}^2 N \to \infty$ to obtain a setup where R (the curvature scale of the solution) is much larger than l_s and the classical supergravity solution can be trusted.