Lecture 6: Symmetries and Invariance Principles – Part II

Overview:

1. Review: Group theory: Example SU(2)
2. SU(3)
3. Discrete symmetries: P, C, T
4. CP violation – K-system
5. CPT

1. Review : Group theory : Example SU(2)

◦ Lie group:

The elements of the group are characterized by a finite number of real parameters a_α with $\alpha = 1, \dots, N$.

$$\boxed{U(\alpha_1, \dots, \alpha_N) = e^{i \sum_{\alpha=1}^{N} a_\alpha \cdot L_\alpha}}$$

Parameters   generators

Define $A = i \sum_{\alpha=1}^{N} a_\alpha \cdot L_\alpha = i\, a_\alpha \cdot L_\alpha$ (Einstein summation convention)

$dA = i L_\alpha\, \delta a_\alpha$

Use:

$$dA = \frac{A}{N} = \frac{i\, a_\alpha \cdot L_\alpha}{N}$$

then:

$$U(\alpha_1, \dots, \alpha_N) = \lim_{N \to \infty} \left[1 + \frac{A}{N} \right]^N = e^{i\, a_\alpha \cdot L_\alpha}$$

"Group is defined by product of infinitesimal transformations around 1"

- commutation relations:

$$\boxed{[L_\alpha, L_\beta] = i\, C_{\alpha\beta\gamma} L_\gamma}$$

The generators and their commutation relations specify a Lie algebra where the $C_{\alpha\beta\gamma}$ are the so-called structure constants.

The generators satisfy the so-called Jacobi identity:

$$\boxed{[L_\alpha, [L_\beta, L_\gamma]] + [L_\beta, [L_\gamma, L_\alpha]] + [L_\gamma, [L_\alpha, L_\beta]] = 0}$$

- There are several spaces that are relevant here:

1. Space on which the generators act: dimension of the respective matrix representation: L_α

2. Space of the group generators: here: N

$$\alpha = 1, \dots, N$$

Simplest possible non-Abelian Lie algebra:

$C_{\alpha\beta\gamma} = \varepsilon_{\alpha\beta\gamma}$ 3 generators

thus:

$$\boxed{[L_\alpha, L_\beta] = i\, \varepsilon_{\alpha\beta\gamma} L_\gamma} \qquad L_\alpha : \quad \alpha = 1, 2, 3$$

now:

$$[L_1, L_2] = i L_3$$
$$[L_2, L_3] = i L_1$$
$$[L_3, L_1] = i L_2$$

Let's fill this with life and choose certain representations:

a) simplest non-trivial representation:

2 x 2 Matrices : Pauli matrices

$$\boxed{L_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad L_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad L_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

$u = e^{i a_\alpha \cdot \sigma_\alpha / 2}$: group elements: special unitary 2x2 matrices

name of group $SU(2)$

Note:

This is the simplest possible representation called fundamental or defining representation.

However:

There will be many other representations, by matrices of various dimensions, different from 2×2.

SU(2):	Has dimensions of:	e.g. spin
	- 1 (the trivial one)	: 0
	- 2 (the fundamental one)	: $\frac{1}{2}$
	- 3, 4, 5, ... (regular or adjoint repr.)	: $1, \frac{3}{2}, 2, \ldots$

chosen depending on your QM-system you study!

su(2) algebra

All representations fulfill the commutation relation:

$$\boxed{[L_\alpha, L_\beta] = i\,\epsilon_{\alpha\beta\gamma}\,L_\gamma}$$

o <u>Special roll of fundamental representations:</u> spin $\frac{1}{2}$

⟶ Build all other representations out of fundamental representations:

$$\boxed{2 \otimes 2 = 3 \oplus 1 \quad \text{or} \quad \frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0}$$

3 dim. repr.   1 dim. repr.

SU(2)

∘ Isospin: SU(2) group

Examples: use isospin to classify:

1. nucleons: n, p $\quad I = 1/2$
2. pions: π^+, π^0, π^- $\quad I = 1$
3. Delta's: $\Delta^{++}, \Delta^+, \Delta^0, \Delta^-$ $\quad I = 3/2$

Increase multiplicity $2I + 1$

fundamental representation of SU(2)

for $I = 1/2$: 2×2 Pauli matrices

How many different I_3 eigenvalues?

States: $\boxed{|I \; I_3\rangle}$ $\qquad I_3 = -I, -I+1, \dots, I-1, I$

I_3: 3rd component of isospin!

$$p = \left|\tfrac{1}{2}\;\tfrac{1}{2}\right\rangle \qquad n = \left|\tfrac{1}{2}\;-\tfrac{1}{2}\right\rangle$$

$$\pi^+ = |1\;1\rangle \qquad \pi^0 = |1\;0\rangle \qquad \pi^- = |1\;-1\rangle$$

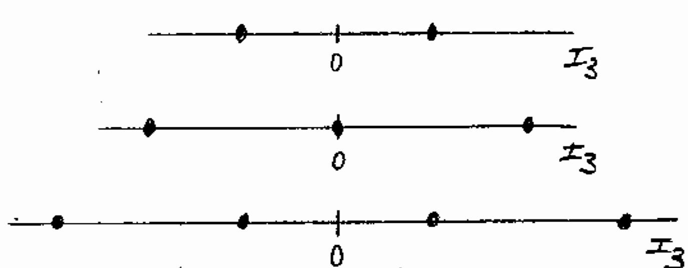
$$\Delta^{++} = \left|\tfrac{3}{2}\;\tfrac{3}{2}\right\rangle \qquad \Delta^{+} = \left|\tfrac{3}{2}\;\tfrac{1}{2}\right\rangle \qquad \Delta^{0} = \left|\tfrac{3}{2}\;-\tfrac{1}{2}\right\rangle \qquad \Delta^{-} = \left|\tfrac{3}{2}\;-\tfrac{3}{2}\right\rangle$$

or:

nucleons: $\qquad I = 1/2$

pions: $\qquad I = 1$

Delta's: $\qquad I = 3/2$



Weight diagrams

2. $su(3)$:

- Lie algebra with respect to $su(3)$:

$$\boxed{[\lambda_i, \lambda_j] = 2i\, f_{ijk}\, \lambda_k} \qquad \lambda_i : i = 1, \dots, 8$$

8 generators

- Fundamental representation of $SU(3)$:

- $SU(2)$ is a sub-group of $SU(3)$

choose λ_1, λ_2 and λ_3 to be the Pauli matrices generalized to three dimensions:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- other 5 generators: $\lambda_4, \dots, \lambda_8$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = \sqrt{\frac{1}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

λ_i: Gell-Mann matrices

Relations among the Gell-Mann matrices and structure constants:

$$\boxed{\begin{aligned} &\mathrm{Tr}\, \lambda_i \lambda_j = 2\, \delta_{ij} \\ &[\lambda_i, \lambda_j] = 2i\, f_{ijk}\, \lambda_k \\ &\{\lambda_i, \lambda_j\} = \lambda_i \lambda_j + \lambda_j \lambda_i = \tfrac{4}{3}\, \delta_{ij} + 2\, d_{ijk}\, \lambda_k \end{aligned}}$$

ijk	f_{ijk}
123	1
147	$\frac{1}{2}$
156	$-\frac{1}{2}$
246	$\frac{1}{2}$
257	$\frac{1}{2}$
345	$\frac{1}{2}$
367	$-\frac{1}{2}$
458	$\frac{\sqrt{3}}{2}$
678	$\frac{\sqrt{3}}{2}$

ijk	d_{ijk}
118	$-\frac{1}{\sqrt{3}}$
146	$\frac{1}{2}$
157	$\frac{1}{2}$
228	$\frac{1}{\sqrt{3}}$
247	$-\frac{1}{2}$
256	$\frac{1}{2}$
338	$\frac{1}{\sqrt{3}}$
344	$\frac{1}{2}$
355	$\frac{1}{2}$
366	$-\frac{1}{2}$
377	$-\frac{1}{2}$
448	$-\frac{1}{2\sqrt{3}}$
558	$-\frac{1}{2\sqrt{3}}$
668	$-\frac{1}{2\sqrt{3}}$
778	$-\frac{1}{2\sqrt{3}}$
888	$-\frac{1}{\sqrt{3}}$

Let us now find the representation for SU(3) and determine the set of simultaneous diagonalization:

$F_i = \frac{1}{2} \lambda_i$ then define:

$$\begin{aligned} I_\pm &= F_1 \pm i F_2 \\ V_\pm &= F_4 \pm i F_5 \\ U_\pm &= F_6 \pm i F_7 \\ I_3 &= F_3 \\ Y &= \frac{2}{\sqrt{3}} F_8 \end{aligned}$$

Eigen values for simultaneous diagonalization:

I_3 , Y and $U_3 = \frac{1}{2}\left(\frac{3}{2}Y - I_3\right)$ and $V_3 = \frac{1}{2}\left(\frac{3}{2}Y + I_3\right)$

can be diagonalized simultaneously:

Eigen values: I_3 , Y, $U_3 = \frac{1}{2}\left(\frac{3}{2}Y - I_3\right)$ $V_3 = \frac{1}{2}\left(\frac{3}{2}Y + I_3\right)$

Represent states of $SU(3)$ by: $|I_3\, Y\rangle$

I_3 : 3rd component of Isospin

Y : Hypercharge

This was introduced by Gell-Mann to classify the mesons and baryons!

Gell-Mann-Nishijima relation:

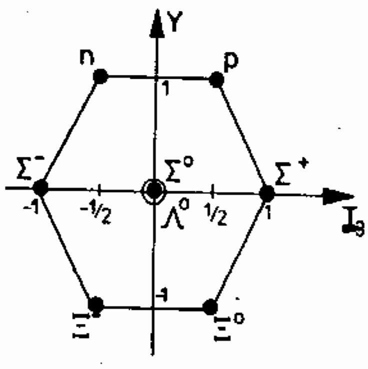
$$\boxed{Q = \tfrac{1}{2}Y + I_3} \qquad Y \to U(1), \quad I_3 \to SU(2) \quad \Big\} \text{ subgroups of } SU(3)$$

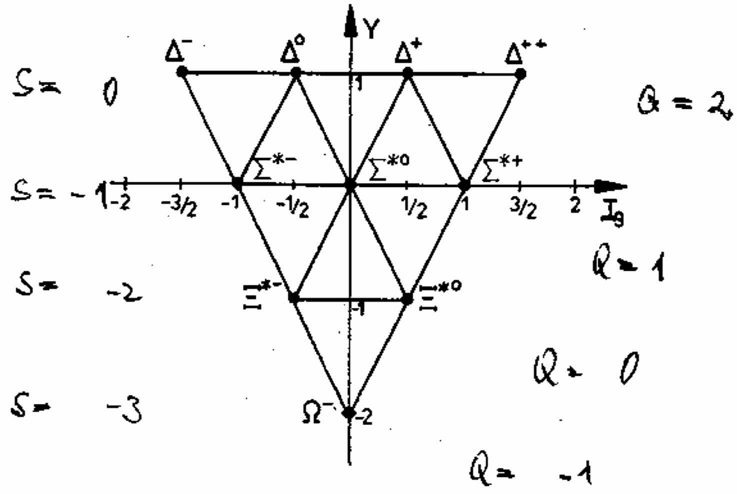
Strangeness: $S = Y - B$ (B: Baryon number)

<u>Examples :</u> <u>Flavor SU(3) :</u>

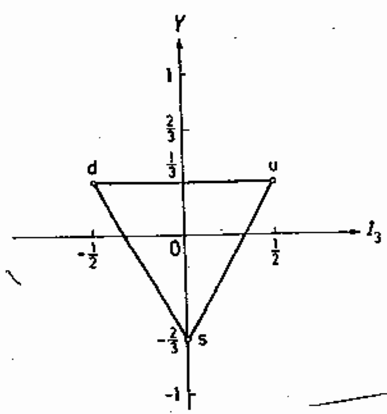
$\longrightarrow$ ("Bad symmetry")

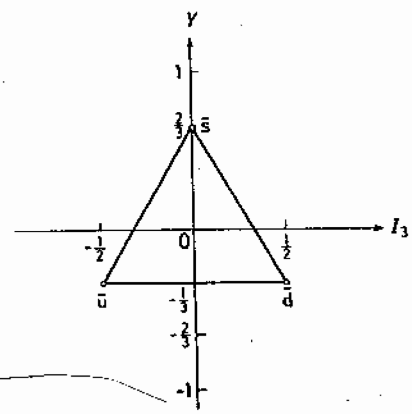
Decompose $| I_3 \, Y \rangle$ . . .





in terms of quark states :





	I	I_3	Y	S	B	Q
u	1/2	+1/2	+1/3	0	1/3	2/3
d	1/2	-1/2	+1/3	0	1/3	-1/3
s	0	0	-2/3	-1	1/3	-1/3

- Color SU(3) : Exact : Fundamental ingredient of QCD

- Three color states:

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \text{ red} \qquad \begin{pmatrix}0\\1\\0\end{pmatrix} \text{ green} \qquad \begin{pmatrix}0\\0\\1\end{pmatrix} \text{ blue}$$

$\longrightarrow$ More details will follow when we discuss QCD!

3. Discrete symmetries:

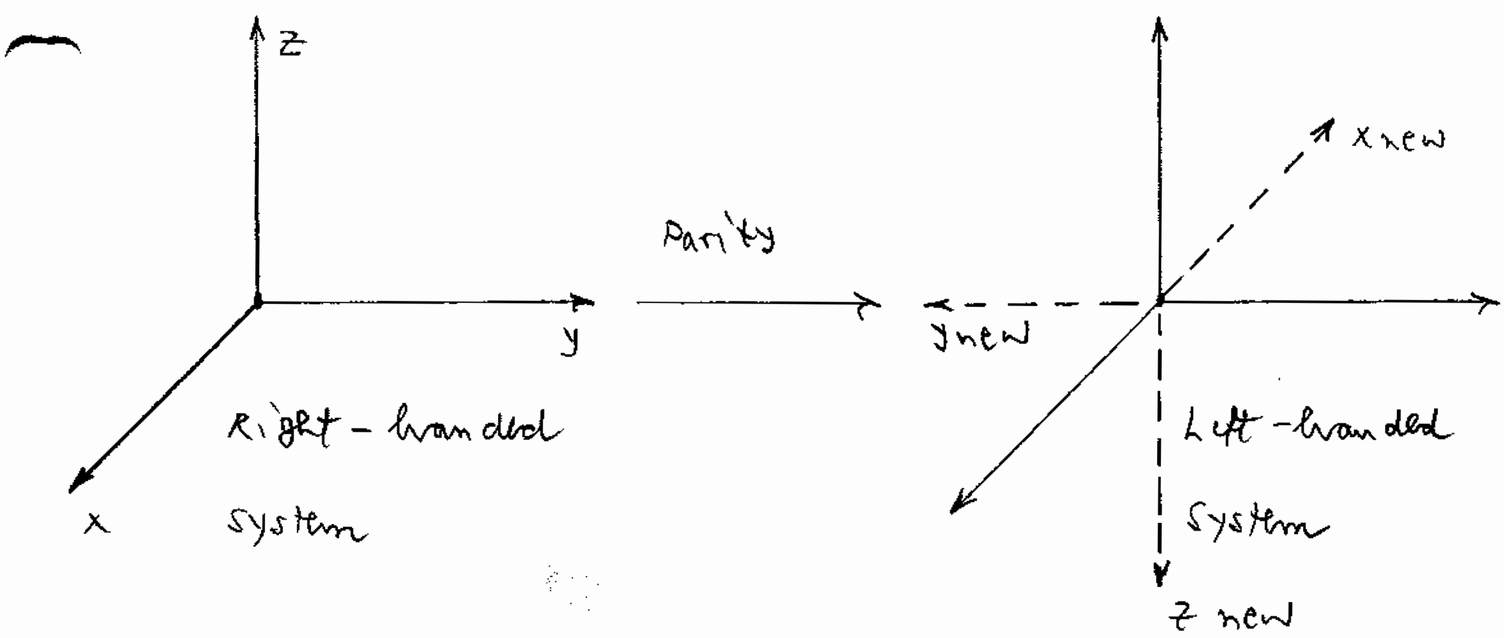
a) Parity:

Parity operation: P

$\longrightarrow$ This denotes an inversion, i.e.:

$$\boxed{\begin{array}{lll} x \to -x & & \\ y \to -y & \text{or} & \vec{r} \to -\vec{r} \\ z \to -z & & \end{array}}$$

What does that mean?



Parity operator: $P\,\psi(\vec{r}) \to \psi(-\vec{r})$

Eigen values : ± 1

obvious: $P^2 = I$

∘ Note:

A wavefunction may or may not have a well-defined parity, which can be even ($P = +1$) or odd ($P = -1$)

Example:

$\psi = \cos x \qquad P\psi \to \cos(-x) = \cos x = \psi$ even

eigenvalue: $+1$

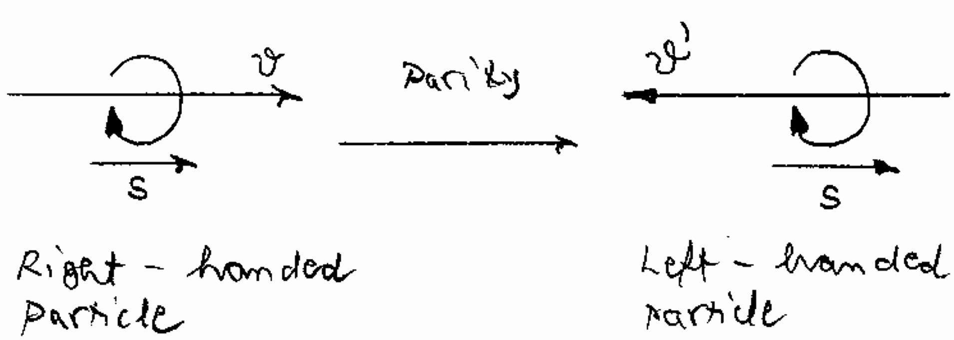
$\psi = \sin x \qquad P\psi \to \sin(-x) = -\sin x = -\psi$ odd

eigenvalue: -1

However: $\psi = \cos x + \sin x$: no definite parity eigenvalue

Consider a particle with a certain spin:

Parity reverses the travel direction without reversing the direction of rotation:



Let us now look at a spherically symmetric potential:

$$H(-\vec{r}) = H(\vec{r}) = H(r) \qquad \text{so that } [P, H] = 0$$

The bound states of the system have definite parity,

e.g. Hydrogen atom:

$$\psi(r, \theta, \phi) = X(r)\; Y_{\ell m}(\theta, \phi)$$
$$= X(r) \sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}} \cdot (-1)^m \cdot P_\ell^m(\cos\theta)\, e^{im\phi}$$

Parity transformation: $\vec{r} \to -\vec{r}$ is equivalent to:

$$\theta \to \pi - \theta \quad : \quad \cos\theta \to -\cos\theta$$
$$\phi \to \phi + \pi \quad : \quad e^{im\phi} \to (-1)^m e^{im\phi}$$
$$r \longrightarrow r$$

With this:
$$Y_{\ell m}(\theta, \phi) \longrightarrow (-1)^\ell \cdot Y_{\ell m}(\theta, \phi)$$

spherical harmonic functions have parity $(-1)^\ell$!

More details: see Sakurai

- Comments :

1. Parity is a multiplicative quantum number in contrast to charge or strangeness which are additive quantum numbers!

2. Behavior of scalars and vectors under parity transformation, P :

	Parity	
scalar	$P(s) = s$	$s = \vec{v}_1 \cdot \vec{v}_2$
pseudoscal	$P(p) = -p$	$p = \vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3)$
V vector (polar vector)	$P(\vec{v}) = -\vec{v}$	
A pseudovector (axial vector)	$P(\vec{a}) = \vec{a}$	

3. Dynamical aspects :

Prior to 1956 it was believed that nature is invariant under parity transformations, i.e. left-handed particles and right-handed take equally part in the fundamental interactions.

Lee and Yang : Experimental evidence for electromag. and strong. interactions !

1956

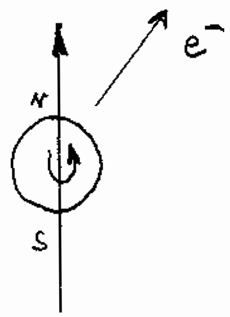
- However to their big surprise:

→ No experimental confirmation in case of weak interactions!

Proposed a test for weak interactions:

→ Experiment by C. S. Wu: β^- decay of ^{60}Co

Align radioactive Cobalt 60 such that their spins pointed for example in the z-direction:



Wu recorded the direction of emitted electrons.

Result: Most electrons came off in a specific direction, preferentially in the direction of N, i.e. in the direction of nuclear spin!

If parity is conserved, e^- should come out in equal proportions of N and S, but:

- Preferred direction: Parity is violated!

<u>comments:</u>

1. Parity is maximally violated
2. Fundamental signature of weak interactions

<u>Consequences for neutrinos:</u>

1. <u>helicity:</u> $+1$ $\qquad\qquad$ -1 $\qquad$ $h = \dfrac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}$

2. For massive particles, i.e. e^-, helicity is not conserved in Lorentz-transformations: Can always choose a system in which the relative alignment of spin and momentum looks different and thus different helicity.

<u>However:</u> neutrinos are massless

It is impossible to reverse the direction of motion of a neutrino by getting into a faster moving reference system!

<u>Thus:</u> <u>helicity of neutrino:</u> Lorentz invariant

The <u>profound consequence of parity violation</u> now is:

All neutrinos are left-handed and

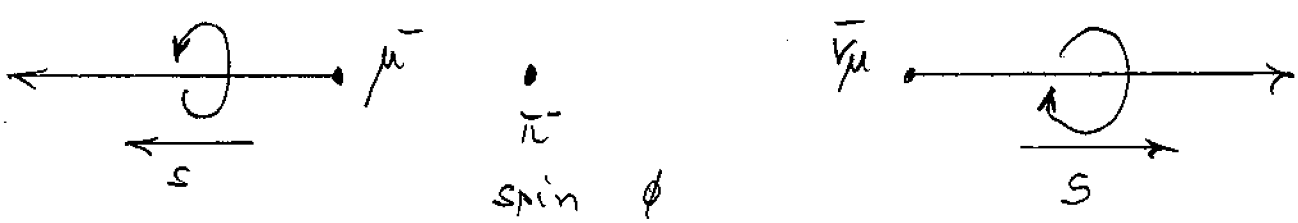
all anti-neutrinos are right-handed !

<u>Before 1956:</u> All neutrinos would be half left-handed and half right-handed

$\longrightarrow$ WRONG : Parity is violated !

This "explains" the preferred the direction of the electron in Wu's experiment.

<u>Indirect method to study the helicity of the neutrino :</u> $\pi^- \rightarrow \mu^- + \bar{\nu}_\mu$ at rest



If the anti-neutrino is right-handed, the muon is right-handed too. This is found experimentally !

- One more parity remark:

- Tau-Theta puzzle:

Two mesons called τ and θ (now known as K^+) appeared to be identical but have two decay modes of different parity:

$$\left.\begin{array}{lll} \theta^+ \to \pi^+ + \pi^0 & & \underline{P = +1} \\ \tau^+ \to \pi^+ + \pi^0 + \pi^0 & & \\ \qquad \pi^+ + \pi^+ + \pi^- & & \underline{P = -1} \end{array}\right\}$$

prompted study of parity properties in weak interactions!

"Same particle" with different parity for underlying processes: Parity is not conserved in weak interactions!

- Fermions: opposite parity for fermions and anti-fermions

- Bosons: same parity for bosons and anti-bosons

Classification of mesons according to their parity structure: $q\bar{q}$ (opposite parity)

$$P = \underbrace{(-1)}_{\substack{\text{different} \\ \text{parity for} \\ q\bar{q}}} \underbrace{(-1)^{l}}_{\substack{\text{orbital} \\ \text{angular} \\ \text{momentum}}} = (-1)^{l+1}$$

We will use this next week when discussing the quark model!

6.) charge conjugation :

classical electrodynamics is invariant under a change of sign of all electric charges!

charge conjugation : "change sign of the charge"

$$\boxed{C|p\rangle = |\bar{p}\rangle}$$

obvious : $C^2 = I$

unlike P, most of the particles in nature are clearly not eigenstates of C!

Assume $|p\rangle$ is an eigenstate of C :

$C|p\rangle = \pm|p\rangle = |\bar{p}\rangle$ therefore: $|p\rangle$ and $|\bar{p}\rangle$ represent the same state!

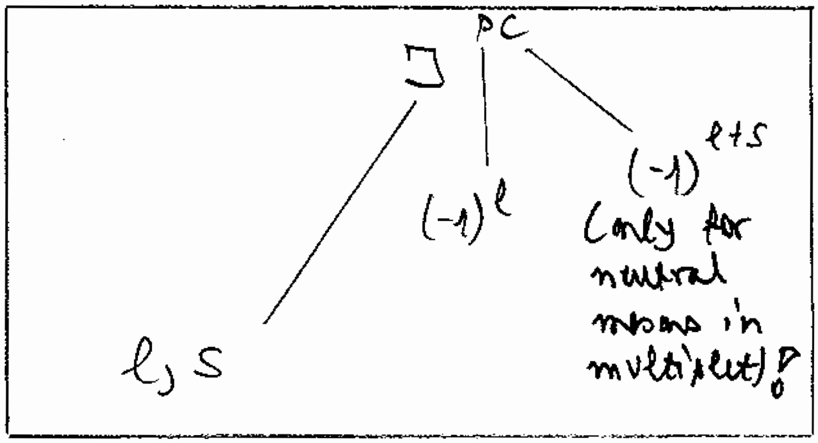
Only particles that are their own anti-particles can be eigenstates of C!

example : π^0, η

It can be shown that a system consisting of a spin-$\frac{1}{2}$ particle and its antiparticle in a configuration with orbital angular momentum l and total spin s constitute an eigenstate of C with the following eigenvalue:

$$(-1)^{l+s}$$

Important for Meson-classification:



* example:

C for π^0: $C = 1$

$\pi^0 \to \gamma + \gamma$

therefore:

System with n photons: $C = (-1)^n$

c) G-parity:

Only a few particles are eigenstates of C. Introduce therefore:

$$\boxed{\text{G-parity } G = (-1)^{I} \cdot C}$$

eigenvalue for a multiplet of isospin I and charge conjugation

example:

- Pion: $I = 1$; $C = +1$ $G = -1$

For n pions in a reaction $G = (-1)^n$

"You can tell how many pions can be emitted in a particular decay"!

ρ^0 meson decay: $J^{PC} = 1^{--}$ $G = 1$

ρ^0 decays to two pions, but not to three!

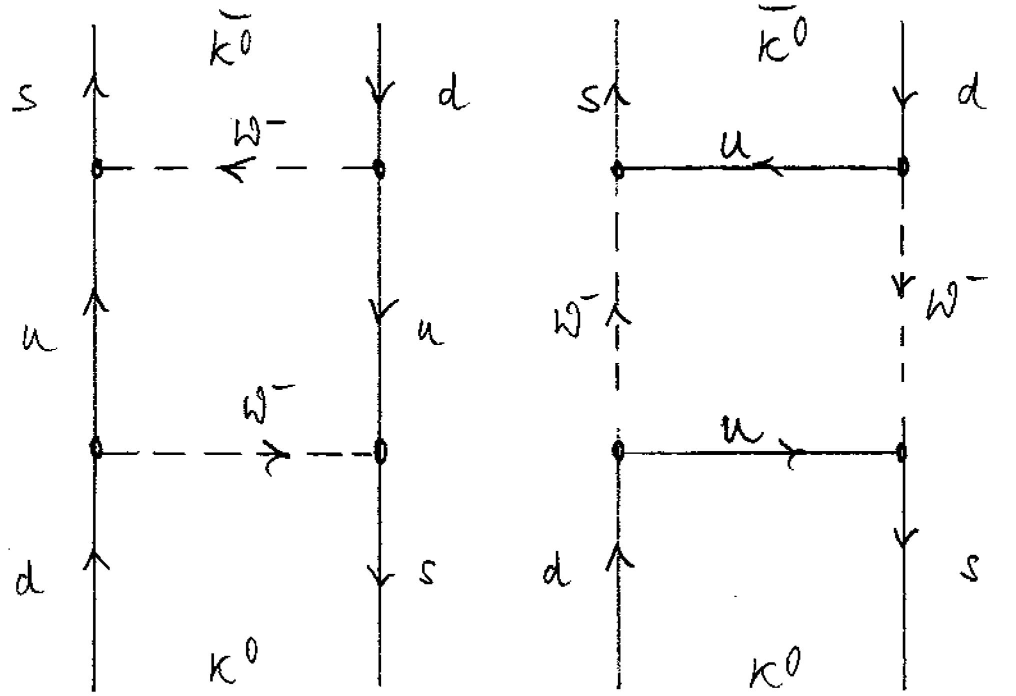
$\rho^0 \to \pi^+ \pi^-$ ρ^0 : $I = 1$

Summary:

1. Strong and electromag. interactions are invariant under C and P

2. Weak interactions are <u>not invariant</u> under C and P!
However, the combination CP <u>"seemed"</u> to to be <u>conserved</u> in <u>weak interactions.</u> How well?

<u>CP invariance:</u> Gell-Mann and Pais



$K^0 \rightleftarrows \bar{K}^0$

mixing

Study this now under C, P and CP operations:

$$P|K^0\rangle = -|K^0\rangle \quad , \quad P|\bar{K}^0\rangle = -|\bar{K}^0\rangle$$

but:

$$C\,|K^0\rangle = |\bar{K}^0\rangle \; , \quad C|\bar{K}^0\rangle = |K^0\rangle$$

$$CP\,|K^0\rangle = -|\bar{K}^0\rangle \quad \text{and} \quad CP\,|\bar{K}^0\rangle = -|K^0\rangle$$

∘ normalized eigenstates of CP:

$$|K_1\rangle = \frac{1}{\sqrt{2}}\left(|K^0\rangle - |\bar{K}^0\rangle\right) \qquad |K_2\rangle = \frac{1}{\sqrt{2}}\left(|K^0\rangle + |\bar{K}^0\rangle\right)$$

With:

$$CP\,|K_1\rangle = |K_1\rangle \qquad CP\,|K_2\rangle = -|K_2\rangle$$

If CP is conserved in weak interactions, K_1 can only decay into a state with CP = +1, whereas K_2, can only go to a state with CP = -1.

Is this so in nature?

$m_{K_1} - 2m_\pi = 220\,\text{MeV}$ $\qquad$ $m_{K_2} - 3m_\pi = 80\,\text{MeV}$

$K_1 \to 2\pi$ (faster decay)	$K_2 \to 3\pi$ (slower decay)

$\ell = \beta\gamma c\tau$

$\tau_1 = 0.89 \cdot 10^{-10}\,s$ $\qquad$ $\tau_2 = 5.2 \cdot 10^{-8}\,s$

$c\tau = 2.7\,cm$ $\qquad$ $c\tau = 15.5\,cm$

$P = (-1)^{\ell+1}$

$C = (-1)^{\ell+S}$

Prepare a beam with K_1 and K_2:

$\pi: \ \ell = S = 0$

$$|K^0\rangle = \frac{1}{\sqrt{2}}\left(|K_1\rangle + |K_2\rangle\right)$$

CP : -1

Will decay away ... $\qquad$ Will be left with K_2's down the beam line at the experiment

Short $\qquad$ Long

- Define the long contribution:

$$\boxed{|K_L\rangle = \frac{1}{(1+|\varepsilon|^2)^{1/2}}\left(|K_2\rangle + \varepsilon|K_1\rangle\right)}$$

If CP is conserved, $\varepsilon = 0$, should only see K_2's and no K_1's!

Cronin, Fitch (1964): Found 45 2π decays at the end of a beam 57 feet long out of 22,700 decays!

- CP is violated : <u>Small effect</u>!

not "easy" to theoretically really understand, compared to P violation!

<u>Subsequent experiment:</u>

- 34% of K_L's decay through 3π mode
- <u>39% go to:</u>

a.) $\pi^+ + e^- + \bar{\nu}_e$

b.) $\pi^- + e^+ + \nu_e$

CP

<u>If CP is conserved, (a) and (b) should occur in equal proportions, but:</u>

K_L decays more often into positrons than electrons!

<u>CP violation:</u> Unequal treatment of particles and anti-particles suggests it may be responsible for the dominance of matter over antimatter in the universe!

◦ CPT theorem:

important principle of quantum field theory. All interactions are invariant under the combined operation of C, P and T taken in any order.

Implications:

1. particles and anti-particles should have the same mass and life time
2. Particles and anti-particles have magn. moment equal in magnitude, but opposite in sign

Since CP is violated it is expected that there is also violation in T to preserve CPT.

⟶ important to look for T violation (no experimental evidence for T violation so far!)

Tests of *CPT* theorem

		Limit on fractional difference
Lifetime	$\tau_{\pi^+} - \tau_{\pi^-}$	$< 10^{-3}$
	$\tau_{\mu^+} - \tau_{\mu^-}$	$< 10^{-4}$
	$\tau_{K^+} - \tau_{K^-}$	$< 10^{-3}$
Magnetic moment	$\lvert\mu_{\mu^+}\rvert - \lvert\mu_{\mu^-}\rvert$	$< 10^{-8}$
	$\lvert\mu_{e^+}\rvert - \lvert\mu_{e^-}\rvert$	$< 10^{-10}$
Mass	$M_{\pi^+} - M_{\pi^-}$	$< 10^{-3}$
	$M_{\bar{p}} - M_p$	$< 10^{-4}$
	$M_{K^+} - M_{K^-}$	$< 10^{-4}$
	$M_{K^0} - M_{\bar{K}^0}$	$< 10^{-14}$