8.325 Homework 2

Iain Stewart, Feb. 23, 2007

Due: In lecture March 8.

Problem 1) MS beta-function with multiple dimensionful couplings

Consider a field theory with a set of couplings $g_1, g_2, \ldots, g_\ell, \ldots$ which we'll call $\vec{g}$. Your goal is to derive an expression for the beta-functions in the MS-scheme that is valid at any order in perturbation theory. (It will also be valid for both renormalizable theories and "non-renormalizabile" theories with irrelevant operators.) Let $\Delta_l(d) = \Delta_\ell + \epsilon\,\rho_\ell$ be the dimension of the bare coupling $g_\ell^{\rm bare}$ in $d = 4-2\epsilon$ dimensions. We can define a dimensionless running coupling in the MS-scheme by

$$g_\ell^{\rm bare}\mu^{-\Delta_\ell(d)} = g_\ell(\mu,d) Z_{g_\ell}(\vec{g}),$$

where $g_\ell(\mu,d)$ is analytic in d and Z_{g_ℓ} is a series in $1/\epsilon$. Prove that the beta-function for $g_\ell \equiv g_\ell(\mu,d)$ is

$$\beta(g_\ell,d) = \mu\frac{d}{d\mu}g_\ell = -\epsilon\rho_\ell\, g_\ell - \Delta_\ell\, g_\ell + g_\ell\sum_m \frac{da_1^\ell(\vec{g})}{dg_m}\rho_m g_m\,, \qquad (1)$$

where a_1^ℓ is the coefficient of the $1/\epsilon$ pole term in Z_{g_ℓ}. Find a recursion relation for the coefficients of the higher poles, $a_k(\vec{g})$. How are these results modified in the $\overline{\rm MS}$-scheme? [Side Remark: in this notation the standard beta-function is $\beta(g_\ell) = \beta(g_\ell, 4)$.]

Problem 2) Scheme and gauge dependence of beta-functions

Consider a renormalizable non-abelian gauge theory. Let $g(\mu_R)$ and $\beta(g)$ be the renormalized coupling and beta-function in a mass-independent renormalization scheme. This could be the MS-scheme or an offshell momentum subtraction scheme, etc.

a) Prove that in any scheme the first term in the series expansion of $\beta(g)$ is gauge independent.

b) Prove that the first two terms in the series expansion of $\beta(g)$ are scheme independent.

c) Finally, prove that in the MS-scheme the renormalized coupling is gauge independent. Thus in this scheme all terms in $\beta(g)$ are gauge independent. Use this result to strengthen the statement in a).

Problem 3) QCD beta-function in background field gauge

Using background field gauge derive the lowest order beta-function for QCD in a massless scheme by carrying out the computation of the 3 diagrams discussed in lecture. Express your result in terms of the quadratic adjoint Casmir C_A and the number of quark flavors n_f. What is the beta-function for an SU(2) gauge theory with 6 flavors?

Problem 4) The QCD running coupling and thresholds

We originally motivated the discussion of mass-independent renormalization schemes by considering $\mu \gg m$. However, these schemes are well defined regardless of the relation between μ and m, and so we can consider using them for $\mu \simeq m$ and for that matter $\mu \leq m$. In this problem we'll explore how this works in QCD at one-loop order.

In lecture you saw that in a mass-dependent scheme for QED the electron "decouples" from the beta-function as we go below its mass scale, falling off quite rapidly, $\beta(\mu \ll m) \sim \mu^2/m^2$. However, if you consider the mass-independent QED beta-function with an e^-, μ^-, and τ^- then apriori you have the same beta-function for $m_e \ll \mu \ll m_\mu$, $\mu \gg m_\tau$, or any other value of μ. The issue with a mass-independent scheme is that $\alpha(\mu)$ is not smart enough to know that heavy particles in the field theory should decouple. This is an important piece of physics that we're going to build into the mass-independent schemes by hand. To do this consider evolving the coupling down from a $\mu \gg m_\tau$ with the beta-function with 3-leptons, $n_\ell = 3$. When we reach $\mu = m_\tau$ we'll decree that the tau is removed from our theory, so that below this scale we switch to using a beta function with $n_\ell = 2$. Lets call the coupling in the theory with n_ℓ leptons $\alpha^{(n_\ell)}(\mu)$. To ensure that this process does not disturb our field theory too much, we'll demand that scattering amplitudes computed in the theory with $n_\ell = 3$ and $n_\ell = 2$ are the same at $\mu = m_\tau$. At lowest order in perturbation theory this just implies continuity of the coupling at the boundary, $\alpha^{(3)}(m_\tau) = \alpha^{(2)}(m_\tau)$. At each mass-threshold we'll repeat the above procedure to build in the decoupling by hand.[1]

Lets apply the same logic to QCD to give couplings $\alpha_s^{(n_f)}(\mu)$ which satisfy the mass-independent beta-function equation for n_f-flavors. From $\alpha_s^{(n_f)}(\mu)$ we can define the integration constant $\Lambda_{\rm QCD}^{(n_f)}$.

a) Let $\alpha_s^{(5)}(m_Z) = 0.118$ with the physical Z-boson mass be your initial condition. Lets take the mass of the bottom and charm quarks to be $m_b = 5\,\text{GeV}$ and $m_c = 2\,\text{GeV}$ (slightly heavier than in nature). Using the decoupling procedure described above, compute $\alpha_s^{(3)}(\mu = 1.5\,\text{GeV})$. Compare it numerically to $\alpha_s^{(5)}(\mu = 1.5\,\text{GeV})$.

b) Consider QCD with $m_u = m_d = m_s = 0$ and note that in this theory the proton-mass is dominated by non-perturbative dynamics of these three quarks. Hence we expect the proton mass $m_p \propto \Lambda_{\rm QCD}^{(3)}$. Derive a relation between $\Lambda_{\rm QCD}^{(6)}$ and $\Lambda_{\rm QCD}^{(3)}$ that only involves heavy-quark masses. Now imagine that the strong coupling is fixed at some very high scale (eg. at a unification scale $\sim 10^{16}\,\text{GeV}$), and predict how much the proton-mass changes if you double the b-quark mass.

[1] In the context of effective field theory this procedure of removing particles is known as "integrating out" a massive degree of freedom, and ensuring the continuity of the S-matrix elements is known as "matching".