Chapter 5

Anomalies

When classical symmetry can not be maintained in QFT, we say we have an anomaly. Note, this is quite different from spontenous symmetry breaking.

We are just discussed one type, anomaly in scale invariance. Mass without mass (dimensional transmutation). PQCD unification scale/renormalization.

Another kind, with a rather difficult flavor is connected with chiral symmetry. Many applications:

- Eliminate extra $U_A(1)$ symmetry of massless QCD (approximate for real QCD, but still too good).
- Eliminate I_3^S (axial isospin) of QCD $\times$ QCD $\Rightarrow m_{n'} >> m_\pi$ (actually, modify it) $\pi^\circ \rightarrow 2\gamma$
- Constraint on what QFTs are consistent by demanding no anomalies in gauge symmetries.
- Connections with topology/solitons.
- $\cdots$
- Hawking radiation (recent work).

Begin with a very concrete low basis approach. $\overset{A}{\Delta}\ V\ V$ graph for mass fermions.

$$I^{\lambda\mu\nu} \equiv i\int \frac{d^4p}{(2\pi)^4} tr(\gamma^\lambda\gamma^5 \frac{1}{\not p - \not q}\gamma^\nu \frac{1}{\not p - \not k_1}\gamma^\mu \frac{1}{\not p} + crossed) \tag{5.1}$$

Check if $k_1\mu I^{\lambda\mu\nu} \stackrel{?}{=} 0$

$$\frac{1}{\not p - \not k_1}\not k_1 \frac{1}{\not p} \quad = \quad \frac{1}{\not p - \not k_1} - \frac{1}{\not p} \tag{5.2}$$

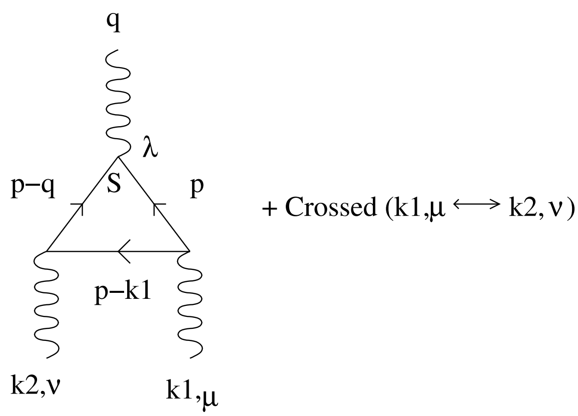

Figure 5.1: Low Basis.

$$\frac{1}{\not p - \not q}(\not q - \not k_2)\frac{1}{\not p - \not k_2} = \frac{1}{\not p - \not q} - \frac{1}{\not p - \not k_2} \tag{5.3}$$

$$k_{1\mu} I^{\lambda\mu\nu} = i\int \frac{d^4p}{(2\pi)^4} tr[\gamma^\lambda\gamma^5(\frac{1}{\not p - \not q}\gamma^\nu\frac{1}{\not p - \not k_1} - \frac{1}{\not p - \not k_2}\gamma^\nu\frac{1}{\not p})] \tag{5.4}$$

This is superficially linearly divergent (note $\sum pp \to 0$ in numerator). Formally, $p \to p - k_1$ in 2^{nd} term makes them cancel. But, linear divergence is dangerous – shifts can leave finite surface terms at $|p| \to \infty$.

Careful shift:

$$f(p+a) \approx \underbrace{f(p)}_{cancels} + \underbrace{a^\mu \frac{\partial}{\partial p^\mu} f(p)}_{finite\ \int} + \underbrace{higher s}_{\to\infty} \tag{5.5}$$

$$\int d^4p[f(p+a) - f(p)] = \lim_{p\to\infty}\int \underbrace{d\Omega}_{angular\ average} \frac{p_\mu}{p} f(p) 2\pi^2 p^3 \underbrace{ia^\mu}_{from\ euclidean\ rotation} \tag{5.6}$$

$$f(p) = tr\gamma^\lambda\gamma^5\frac{1}{p-k_2}\gamma^\nu\frac{1}{p} = \frac{4i\epsilon^{\tau\nu\sigma\lambda}k_{2\tau}p_\sigma}{(p-k_2)^2p^2} \tag{5.7}$$

$$a^\mu = k_1^\mu \tag{5.8}$$

$$k_{1\mu}I^{\lambda\mu\nu} = \lim_{p\to\infty} d\Omega \frac{4i\epsilon^{\tau\nu\sigma\lambda}k_{2\tau}(p_\sigma p_{\mu i})}{p^4(2\pi)^2} 2\pi^2 p^3 i k_1^\mu$$

$$= \frac{i}{8\pi^2}\epsilon^{\lambda\nu\tau\sigma}k_{1\tau}k_{2\sigma}\ (\neq 0) \tag{5.9}$$

However the whole calculation is bogous because we could shift by anything in internal momentum.

$$I^{\lambda\mu\nu}(p \to p + a, k_1, k_2) - I^{\lambda\mu\nu}(k_1, k_2) = \frac{i}{8\pi^2}\epsilon^{\sigma\nu\mu\lambda}a_\sigma + crossed \tag{5.10}$$

e.g., choosing $a_\sigma = \frac{1}{2}(k_2 - k_1)$ gives conceived V, (or CVC). This corresponds to symmetric integration.

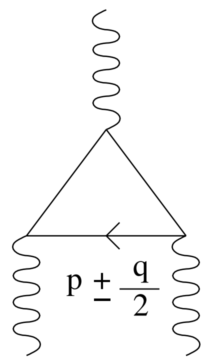

Figure 5.2: Symmetric Integration.

With this choice, though, the naive axial divergence gets doubled, just cancelled.

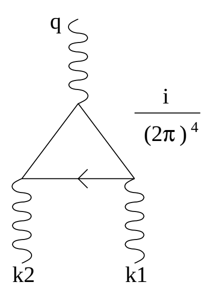

Figure 5.3: Axial Divergence.

$$\frac{1}{\not p}\not q\gamma_5\frac{1}{\not p - \not q} = \frac{1}{\not p}(\not p - (\not p - \not q))\gamma_5\frac{1}{\not p - \not q} = \gamma_5\frac{1}{\not p - \not q} + \frac{1}{\not p}\gamma_5 \tag{5.11}$$

As before, but with $\lambda \leftrightarrow \mu$

$$\begin{aligned} \frac{i}{(2\pi)^4}\int\frac{d^4p}{(2\pi)^4}tr\gamma_5\frac{1}{p-q}\gamma_\nu\frac{1}{p-k_1}\gamma_\mu &= k_{1\mu}[\epsilon^{\mu\nu\lambda\sigma}k_{2\sigma}] \\ &= -\underbrace{(-(k_1+k_2)_\lambda)}_{q_\lambda}[\epsilon^{\mu\nu\lambda\sigma}k_{2\sigma}] \end{aligned} \tag{5.12}$$

This leads to

$$\partial_\mu J^{\mu S} = \frac{e^2}{(4\pi)^2}\epsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma} \tag{5.13}$$

2 fields, 4 terms $\Rightarrow$ factor γ.

Not a very satisfactory derivation, of course. Much better is to use Pauli-Villars regulator, add massive spin $\frac{1}{2}$ boson for opposite sign in loop. Take $M \to \infty$ at the end, because we do not want this in the physical spectrum.

Now we can shift with a clean conscience

$$tr\gamma^\lambda\gamma_5\frac{1}{\not{p}-\not{q}-M}\gamma^\nu\frac{1}{\not{p}-\not{k}_1-M}\gamma^\mu\frac{1}{\not{p}-M} \tag{5.14}$$

where $\not{k}_1 = (\not{p}-M)-(\not{p}-\not{k}_1-M)$, vector is no problem.

Axial vector

$$\begin{aligned}
\not{q} &= (\not{p}-M)-(\not{p}-\not{q}-M) && (5.15)\\
\frac{1}{\not{p}-M}\not{q}\gamma_5\frac{1}{\not{p}-\not{q}-M} &= \gamma_5\frac{1}{\not{p}-\not{q}-M}+\frac{1}{\not{p}-M}\gamma_5\frac{\not{p}-\not{q}+M}{\not{p}-\not{q}-M} \\
&= odd\ term + 2M\frac{1}{\not{p}-M}\gamma_5\frac{1}{\not{p}+\not{q}-M} && (5.16)\\
answer &\propto M\int d^4ptr\gamma^5\frac{1}{\not{p}-\not{q}-M}\gamma^\nu\frac{1}{\not{p}-\not{k}_1-M}\gamma^\mu\frac{1}{\not{p}-M} \\
sim\ & \frac{M\epsilon^{\mu\nu\lambda\sigma}(p_{\to 0},k_1,k_2)}{(p^2-M^2)^3}\ (finite\ as\ M\to\infty) && (5.17)
\end{aligned}$$

This gives the same final answer, of course.