Physics 8.321, Fall 2002
Homework #4

Due **Wednesday, October 9** by 4:30 PM in the 8.321 homework box in 4-339B.

1. Sakurai: Problem 21, Chapter 1 (page 64)

2. Sakurai: Problem 22, Chapter 1 (page 64)

3. Let $H = \frac{p^2}{2m} + V(x)$ be the Hamiltonian for a one-dimensional quantum system with discrete eigenstates $H|a\rangle = E_a|a\rangle$. Show the following results:

(a) $\sum_{a'} |\langle a|x|a'\rangle|^2 (E_{a'} - E_a) = \frac{\hbar^2}{2m}$.

(b) $\langle a|p|a'\rangle = \frac{im}{\hbar}(E_a - E_{a'})\langle a|x|a'\rangle$
and hence $\sum_{a'} |\langle a|x|a'\rangle|^2 (E_{a'} - E_a)^2 = \frac{\hbar^2}{m^2}\langle a|p^2|a\rangle$.

(c) Generalize to 3 dimensions and show the quantum virial theorem
$\langle a|\frac{p^2}{2m}|a\rangle = \frac{1}{2}\langle a|\mathbf{x}\cdot\boldsymbol{\nabla}V(\mathbf{x})|a\rangle$.

4. A particle of mass m is in a 1D potential $V(x) = v\delta(x-a) + v\delta(x+a)$ where $v < 0$.

(a) Find the wave function for a bound state with even parity ($\psi(x) = \psi(-x)$).

(b) Find an expression for the energy for even parity states, and determine how many such states exist.

(c) Solve for the even parity bound state energy when $\frac{ma|v|}{\hbar^2} \ll 1$.

(d) Repeat parts (a) and (b) for odd parity ($\psi(x) = -\psi(-x)$). For what values of v are there bound states?

(e) Find the even and odd parity state binding energies for $\frac{ma|v|}{\hbar^2} \gg 1$, and explain physically why these energies move closer together as $a \to \infty$

5. Define the *coherent state* $|\phi\rangle = e^{\phi a^\dagger}|0\rangle$, where ϕ is a complex number, $a^\dagger$ is the creation operator for a harmonic oscillator, and $|0\rangle$ is the oscillator ground state. Show that $|\phi\rangle$ has the following properties:

(a) $|\phi\rangle = \sum \frac{\phi^n}{\sqrt{n!}}|n\rangle$

(b) $a|\phi\rangle = \phi|\phi\rangle$

(c) $\langle\phi|\phi'\rangle = e^{\phi^*\phi'}$

(d) $\langle\phi| : A(a^\dagger, a) : |\phi'\rangle = e^{\phi^*\phi'} A(\phi^*, \phi')$,
where $: A(a^\dagger, a) :$ is "normal ordered" so that all creation operators $a^\dagger$ are to the left of all annihilation operators a.

(e) $\int \frac{d\phi^* d\phi}{2\pi i} e^{-\phi^*\phi}|\phi\rangle\langle\phi| = \mathbb{1}$. (completeness for coherent states)

6. Define a *squeezed state* to be a state of the form

$$|\alpha, \beta, \gamma\rangle = e^{\alpha + \beta a^\dagger + \gamma (a^\dagger)^2} |0\rangle \tag{1}$$

in the single harmonic oscillator Hilbert space

(a) Compute the norm $\langle \alpha, \beta, \gamma | \alpha, \beta, \gamma \rangle$ in the special case $\beta = 0$. What is the condition needed for this norm to be finite? Can you generalize your result to $\beta \neq 0$?

(b) Show that the position basis state $|x'\rangle$ can be written in the form (1), and find the associated values $\alpha(x'), \beta(x'), \gamma(x')$. Does your expression for $|x'\rangle$ give a state of finite norm in the Hilbert space?

(c) Use your answer to (b) to give squeezed state descriptions of the kets associated with the wavefunctions $\psi(x') = \delta(x')$ and $\psi(x') = 1$.

(d) Describe the kets associated with the wavefunctions $\delta(x' \pm y')$ in squeezed state form

$$\exp\left[F(a_x^\dagger, a_y^\dagger)\right] (|0\rangle_x \otimes |0\rangle_y)$$

where F is a quadratic function of $a_x^\dagger, a_y^\dagger$.