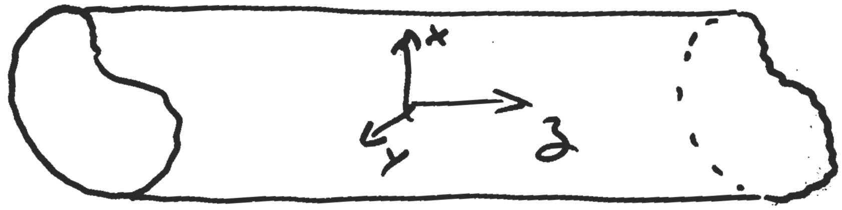
Wave guides

Cylinder shape:



$$\vec{E}(x,y,z,t) = \vec{E}(x,y)\, e^{ikz - i\omega t}$$

Same for $\vec{B}(x,y,z,t)$

Maxwell eqs.

$$\nabla \times E = \frac{i\omega}{c} B \qquad \nabla \cdot E = \nabla \cdot B = 0$$

$$\nabla \times B = -\frac{i\omega}{c} E$$

Wave eqn. $\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\begin{Bmatrix}E\\B\end{Bmatrix} = 0 \to \left(\nabla^2 + \frac{\omega^2}{c^2}\right)\begin{Bmatrix}E\\B\end{Bmatrix} =$

Boundary conditions:

$$E_{\parallel} = \zeta\,(n \times H_{\parallel})$$

e.g., for a conductor

$$\zeta = (i-1)\sqrt{\frac{\omega}{8\pi\sigma}}$$

$$\lambda \begin{Bmatrix}E\\B\end{Bmatrix} = \left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right\}\begin{Bmatrix}E\\B\end{Bmatrix}$$

$$\lambda = k^2 - \frac{\omega^2}{c^2}$$

Boundary value problem:

gives relation between ω and k

$\omega_i(k)$ — k is wave vector

ω is frequency

i labels modes

Solving for the modes:

$$E = \vec{E}_t + E_z \qquad B = \vec{B}_t + B_z$$

normal to $\hat{z}$

Maxwell eqs:

$$\frac{\partial \vec{E}_t}{\partial z} + \frac{i\omega}{c}(\hat{z} \times B_t) = \vec{\nabla}_t E_z \qquad \nabla_t \times E_t = \frac{i\omega}{c} B_z \hat{z}$$

$$* \quad \frac{\partial \vec{B}_t}{\partial z} - \frac{i\omega}{c}(\hat{z} \times E_t) = \vec{\nabla}_t B_z \qquad \nabla_t \times B_t = -\frac{i\omega}{c} E_z \hat{z}$$

$$\nabla_t \cdot E_t = -\frac{\partial E_z}{\partial z} \qquad \nabla_t \cdot B_t = -\frac{\partial B_z}{\partial z}$$

Special solution (TEM mode)

$$E_z = B_z = 0$$

$$\downarrow$$

$$k = \frac{\omega}{c} \quad \text{and} \quad \nabla_t \times E_{TEM} = 0 \quad \text{(Same for B)}$$

$$\nabla \cdot E_{TEM} = 0$$

perfect conductor: $E_{\parallel} = 0$, $B_{\perp} = 0$ @ surface

The problem for E_{TEM} is identical to electrostatics

Thus: No TEM mode in a hollow tube

TEM exist for coaxial cables

General modes; two types TE & TM
transverse electric (magnet

Boundary conditions $n \times E = n \cdot B = 0$
perfect conductor

$E_z = 0$ @ surface

$\frac{\partial B_z}{\partial n} = 0$ @ surface (from eq. *)

<u>TE mode:</u> $E_z = 0$

$$\begin{cases} \nabla_t^2 B_z = \lambda B_z & \lambda = k^2 - \frac{\omega^2}{c^2} \\ \frac{\partial B_z}{\partial n} = 0 & \text{@ surface} \end{cases}$$

<u>TM mode:</u> $B_z = 0$

$$\begin{cases} \nabla_t^2 E_z = \lambda E_z & \lambda = k^2 - \frac{\omega^2}{c^2} \\ E_z = 0 & \text{@ surface} \end{cases}$$

Determine $\lambda_i = -\frac{\omega_i^2}{c^2} \rightarrow \omega^2 = \omega_i^2 + k^2 c^2$

ω_i are <u>cutoff frequencies</u>

contrast with TEM ($\omega_i = 0$)

Another method

$$B_t = \pm Z^{-1}\, \hat{z} \times E_t \qquad Z = \frac{ck}{\omega} \text{ (TM)}$$

propagation direction

$$Z = \frac{\omega}{ck} \text{ (TE)}$$

TM: $\vec{E}_t = \pm \dfrac{ik}{\gamma^2} \nabla_t \psi$

TE: $\vec{B}_t = \pm \dfrac{ik}{\gamma^2} \nabla_t \psi$

$$\psi e^{\pm ikz} = \begin{cases} E_z \\ B_z \end{cases}$$

$$\nabla_t^2 \psi + \gamma^2 \psi = 0 \qquad \gamma^2 = \frac{\omega^2}{c^2} - k^2$$

TM: $\psi = 0$

@ surface

TE: $\dfrac{\partial \psi}{\partial n} = 0$

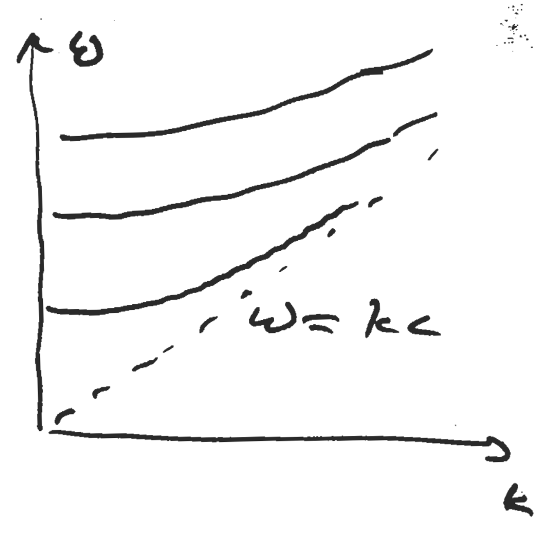
$$\boxed{\text{Set: } \gamma_i = \frac{\omega_i}{c}}$$

phase velocity:

$$v_{ph} = \frac{\omega}{k} > c$$

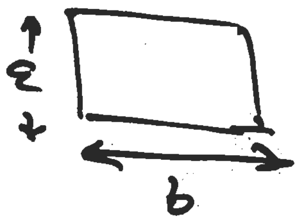
group velocity:

$$v_g = \frac{d\omega}{dk} < c$$



Rectangular wave guide

TE: $\frac{\partial \psi}{\partial n} = 0$ @ surface



$$\psi_{mn} = H_0 \cos \frac{\pi m x}{a} \cos \frac{\pi n y}{b}$$

m, n positive integers (or zero)

$$\omega = \pi c \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^{\frac{1}{2}}$$

TM: (the same)

$\psi = 0$ @ surface

$$\psi_{mn} = E_0 \sin \frac{\pi m x}{a} \sin \frac{\pi n y}{b}$$

m, n positive integers

$$\omega = \pi c \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^{\frac{1}{2}}$$