Lecture (3)

Today:

- Principle of Least Action
- Euler-Lagrange Equations

For tomorrow

1. read LL 1-5 again (really!)
2. do pset problems 7-9

1 Principle of Least Action (PLA)

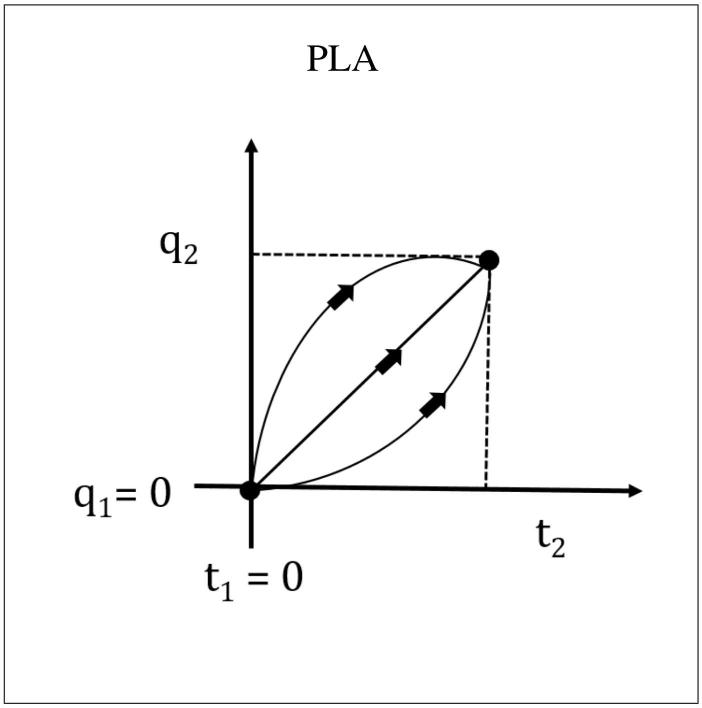
Principle of Least Action (PLA):
for some $L(q, \dot{q}, t)$, the motion of a system minimizes
$S = \int_{t_1}^{t_2} L\, dt$, where $S =$ "action"
for a given $q(t_1)$ *and* $q(t_2)$

S is the action, and L is the Lagrangian. In the most general case L need not be $T - U$. But in most interesting cases, $L = T - U$.

Let's look at some simple examples.

Free Particle in 1D

$$\Rightarrow U = 0 \Rightarrow L = \frac{1}{2} m \dot{q}^2$$



The PLA is not like Newtonian thinking. You assume that you KNOW THE END POINTS, and ask what happened in between. With $F = ma$, you assume you know the initial position AND VELOCITY, and then move forward in time.

For the PLA any *trial path* is valid. The one with minimal S is the *true path*. For this example, I'll consider parabolic paths.

Constant Velocity Path

$$t_1 = 0\ ,\ q(t_1) = q_1 = 0$$
$$q(t) = at + bt^2\ ,\ q(t_2) = q_2 = at_2 + bt_2^2$$
$$\Rightarrow a = \frac{q_2}{t_2} - bt_2 = v_2 - bt_2$$
$$\text{where } v_2 = \frac{q_2}{t_2}$$

This leaves us b as a free parameter which we can adjust to find the path with minimum action.

$$S = \int_0^{t_2} \frac{1}{2} m\dot{q}^2 dt \ , \ \dot{q} = a + 2bt$$

$$\text{Mathematica!} \Rightarrow S = \frac{1}{2} m t_2 (v_2^2 + \frac{b^2 t_2^2}{3})$$

$$\text{minimum at } b = 0$$

So, free particles move at constant velocity. Newton's 1st Law! Inertia!
(not really a surprise, I guess...)
Let's try again, but this time with a simple potential.

Simple Potential

$$U = mgq \Rightarrow L = \frac{1}{2} m\dot{q}^2 - mgq$$

$$\Rightarrow S = m \int_0^{t_2} \frac{1}{2}(a + 2bt)^2 - g(at + bt^2) dt$$

$$\Rightarrow S = \frac{1}{2} m t_2 (v_2 (v_2 - g t_2) + \frac{1}{3} t_2^2 (b^2 + bg))$$

$$\frac{\partial S}{\partial b} = 0 \Rightarrow 2b + g = 0 \Rightarrow b = -\frac{g}{2}$$

So we found the parabolic path with minimum action to be the one you would expect from 8.01.

$$a = v_2 - b t_2 = v_2 + \frac{g t_2}{2} = \dot{q}(t = 0) = v_0$$

$$q(t) = v_0 t - \frac{1}{2} g t^2$$

The initial velocity is just what a projectile needs to fly a distance q_2 in time t_2 with acceleration g.

$\Rightarrow$ Projectile motion results from PLA!

Of course the PLA does't say anything about parabolic paths. Any trial path will do! How do you know the *true path*?

The trick is to assume you know the path and then show you are correct by trying to adjust it. (This comes from from the Calculus of Variations, see Marion & Thornton chapter 5 for more info.)

$$S' = \int_{t_1}^{t_2} L(q', \dot{q}', t) dt$$

$$\text{with} \quad \underbrace{q'(t)}_{\text{trial path}} = \underbrace{q(t)}_{\text{true path}} + \underbrace{\eta(t)}_{\text{deviation}} \quad , \quad \Rightarrow \dot{q}' = \dot{q} + \dot{\eta}$$

$$q'(t_1) = q(t_1) \Rightarrow \eta(t_1) = 0$$

$$q'(t_2) = q(t_2) \Rightarrow \eta(t_2) = 0$$

PLA says $S' \approx S$ for small η (first order)

$$\begin{aligned}
S' - S &= \int_{t_1}^{t_2} \underbrace{L(q', \dot{q}', t) - L(q, \dot{q}, t)}_{\delta L} dt \\
L(q', \dot{q}', t) &\approx L(q, \dot{q}, t) + \frac{\partial L}{\partial q}\eta + \frac{\partial L}{\partial \dot{q}}\dot{\eta} \\
\Rightarrow S' - S &\approx \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\eta + \frac{\partial L}{\partial \dot{q}}\dot{\eta} \right) dt \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\eta \right) &= \eta \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) + \frac{\partial L}{\partial \dot{q}}\dot{\eta} \\
\Rightarrow S' - S &\approx \int_{t_1}^{t_2} \eta \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) \right) dt + \underbrace{\int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\eta \right) dt}_{\frac{\partial L}{\partial \dot{q}}\eta\big|_{t_1}^{t_2} = 0} \\
\text{since } \eta(t_1) &= \eta(t_2) = 0
\end{aligned}$$

We are looking for a true path with $S' - S = 0$ for any small deviation $\eta(t)$.

$$S' - S = \int_{t_1}^{t_2} \eta \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right) \, dt = 0$$

for any η

$$\Rightarrow \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \text{ , Euler-Lagrange}$$

And thus we see that the true path must be a solution to the E-L equation! (It is not an accident that this is also the generalization that worked in yesterday's lecture.)

The PLA gives us N second order ODEs for a system with N DOFs. To find the path of our system through our generalized coordinate space, we should provide 2N initial conditions, and solve N 2nd order ODEs. (Mostly, we will keep to N $\in \{1,2,3\}$).

A note about notation: Generally I will write q without the subscript i (as noted yesterday). You can think of this as the 1D case. If you want the ND case, just add i to all of the q's and $\dot{q}$'s. If the expression does not have i as a free index, sum over it. For example, the Euler Lagrange Equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q} \rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

$$\text{or } T = \frac{1}{2}m\dot{q}^2 \rightarrow T = \frac{1}{2}m\sum \dot{q_i}^2$$

$$\text{or } \frac{d}{dt}f(q,t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q}\dot{q} \rightarrow \frac{d}{dt}f = \frac{\partial f}{\partial t} + \sum \frac{\partial f}{\partial q_i}\dot{q}_i$$

(Similar to Einstein summation notation). The same is true for multiple particles:

$$T = \frac{1}{2}m\dot{q}^2 \Rightarrow T = \frac{1}{2}\sum_n m_n \sum_{i_n} \dot{q}_{i_n}^2$$

I will try to avoid the index jungle as much as possible by sticking to 1 particle in 1D when writing equations.

NB:

$$\vec{q} \equiv \{q_i \forall i\} \equiv q$$
$$\text{e.g. } \vec{r} = \{x, y, z\} \text{ or } \{r, \phi, \theta\}$$

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2 Generalized Forces and Momenta

Briefly, here is how we get $F = ma$ from E-L

$$\text{if } L = \frac{1}{2}m\dot{q}^2 - U(q)$$
$$F_i \equiv \frac{\partial L}{\partial q_i} = -\frac{\partial U}{\partial q_i} \text{ since } T \text{ not a function of } q$$
$$\dot{p}_i \equiv \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{d}{dt}(m\dot{q}_i) \text{ since } U \text{ not a function of } \dot{q}$$
$$\text{E-L} \quad \frac{\partial L}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \Rightarrow \vec{F} = \dot{\vec{p}} \text{ (i.e. Newton)}$$

So, while $F = ma$ gets tricky when these conditions are not met, E-L just works; the PLA just got us a very general form of Newton's second law. As such, we will need to give names to the generalizations are of force and momentum that we are used to. They are:

$$F_i \equiv \text{generalized force for coordinate } q_i$$
$$p_i \equiv \text{generalized momentum for velocity } \dot{q}_i$$

Note that the units associated with these generalized forces and momenta may not be what you expect.

Units

$$\begin{aligned}[U] &= \text{energy} = \mathrm{J} = \frac{\text{kg m}^2}{\text{s}^2} \\ [F_i] &= \frac{\text{energy}}{[q_i]} \text{ e.g. } \frac{\text{kg m}}{\text{s}^2} \text{ if } [q_i] = \text{ meters} \\ [p_i] &= \frac{\text{energy} \times \text{time}}{[q_i]} \text{ e.g. } \frac{\text{kg m}}{\text{s}} \text{ if } [q_i] = \text{ meters}\end{aligned}$$

However, since the units of q_i could be anything (e.q. unitless for angles in spherical coordinates) the units of F_i and p_i may be unusual.

3 Math Review

Before we go on to more physics, let's review our mathematical tools.

Chain Rule

$$\frac{\mathrm{d}}{\mathrm{d}x} f(a,b) = \left(\frac{\mathrm{d}a}{\mathrm{d}x}\right)\left(\frac{\partial f}{\partial a}\right) + \left(\frac{\mathrm{d}b}{\mathrm{d}x}\right)\left(\frac{\partial f}{\partial b}\right)$$

Total Derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} f(q,\dot{q},t) = \frac{\partial f}{\partial t} + \dot{q}\frac{\partial f}{\partial q} + \ddot{q}\frac{\partial f}{\partial \dot{q}}$$

Product Rule

$$b\frac{\mathrm{d}a}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(ab) - a\frac{\mathrm{d}b}{\mathrm{d}x}$$

Integration by Parts

$$\int_{x_1}^{x_2} b\,\frac{\mathrm{d}a}{\mathrm{d}x}\,dx = ab|_{x_1}^{x_2} - \int_{x_1}^{x_2} a\,\frac{\mathrm{d}b}{\mathrm{d}x}\,dx$$

We will have q and $\dot{q}$ as the only *implicit* functions of time (i.e. we don't know q and $\dot{q}$ until we solve the equations of motion). We will also generally only have total TIME derivatives. All other are "easy" partial derivatives like $\frac{\partial}{\partial q}$ or $\frac{\partial}{\partial \dot{q}}$.

NB: $\dot{q} = \frac{\mathrm{d}}{\mathrm{d}t} q(t) = \frac{\partial}{\partial t} q(t)$

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4 Lagrangian Workflow

The general workflow for solving problems with Lagrangian Mechanics is:

Lagrangian Workflow:

1. pick generalized coordinates
2. determine $L(q, \dot{q}, t)$
3. compute F_i and $\dot{p}_i$ to find EoM

Finding $L(q, \dot{q}, t)$ requires $T(q, \dot{q})$ and $U(q, t)$
Usually $U(q, t)$ is given. What about $T(q, \dot{q})$?

The Lagrangian formalism is very powerful in that we can pick any coordinate we like, but there is a price to pay: the kinetic energy is complicated.

Kinetic Energy

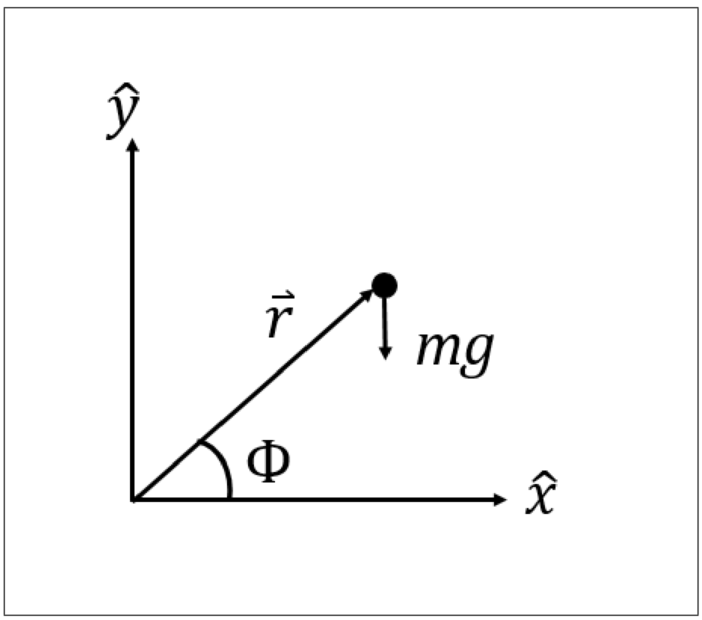
$$\begin{aligned} T &= \frac{1}{2} m \sum_{j,k} a_{jk} \dot{q}_j \dot{q}_k, \text{ for each particle} \\ \text{where } a_{jk} &= \sum_i \frac{\partial r_i}{\partial q_j} \frac{\partial r_i}{\partial q_k}, \ \vec{r} = \{x, y, z\}. \end{aligned}$$

Note that we rely on Cartesian coordinates $\vec{r}$ to find the a_{jk} coefficients (see also Marion & Thornton chapter 6.8, but beware of notational differences).

$$
\begin{aligned}
\text{if } \vec{q} &= \{x, y, z\} \\
a_{xx} &= \left(\frac{\partial x}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2 = 1 \\
a_{xy} &= \frac{\partial x}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial y}{\partial x}\frac{\partial y}{\partial y} + \frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = 0 \\
\Rightarrow T &= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)
\end{aligned}
$$

In Cartesian coordinates this is nothing special, but in others it is tricky. See LL 4.4-4.6.

Let's work through an example to show how all of this machinery works. I will just do 2D projectile motion (e.g. mgh potential), but I will make the unforgivable mistake of using polar coordinates. This will demonstrate the full process in detail, and the value of picking the right coordinates!



2D Projectile Motion

$$
\begin{aligned}
\vec{q} &= \{r, \phi\} \\
x &= r\cos\phi\ ,\ y = r\sin\phi \\
U &= mgy = mgr\sin\phi \\
\frac{\partial x}{\partial r} &= \cos\phi\ ,\ \frac{\partial y}{\partial r} = \sin\phi \\
\frac{\partial x}{\partial \phi} &= -r\sin\phi\ ,\ \frac{\partial y}{\partial \phi} = r\cos\phi
\end{aligned}
$$

In Cartesian Coordinates

$$
\begin{aligned}
T &= \frac{1}{2}m\left((\frac{\partial x}{\partial r})^2(\frac{\partial y}{\partial r})^2\right)\dot{r}^2 + \\
&\quad 2\left(\frac{\partial x}{\partial r}\frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \phi}\right)\dot{r}\dot{\phi} + \left((\frac{\partial x}{\partial \phi})^2 + (\frac{\partial y}{\partial \phi})^2)\dot{\phi}^2\right) \\
T &= \frac{1}{2}m\Big((\cos^2\phi + \sin^2\phi)\dot{r}^2 + \\
&\quad 2(-r\cos\phi\sin\phi + r\sin\phi\cos\phi)\dot{r}\dot{\phi} + \\
&\quad (r^2\sin^2\phi + r^2\cos^2\phi)\dot{\phi}^2\Big) \\
T &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)
\end{aligned}
$$

now we have kinetic energy in 2D polar coordinates

$$
\begin{aligned}
L &= T - U \\
&= \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - mgr\sin\phi
\end{aligned}
$$

from there we find our generalized forces

"Forces"

$$F_r = \frac{\partial L}{\partial r} = r\dot{\phi}^2 - mg\sin\phi \quad \left[\frac{\text{kg m}}{\text{s}^2}\right] \text{ force}$$

$$F_\phi = \frac{\partial L}{\partial \phi} = -mgr\cos\phi \quad \left[\frac{\text{kg m}^2}{\text{s}^2}\right] \text{ torque}$$

and generalized momenta

"Momenta"

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad \left[\frac{\text{kg m}}{\text{s}}\right] \text{ mass x velocity}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} \quad \left[\frac{\text{kg m}^2}{\text{s}}\right] \text{ moment of inertia x angular velocity}$$

finally, the EoM.

Equations of Motion

$$\begin{aligned} F_r = \dot{p}_r \Rightarrow r\dot{\phi}^2 - mg\sin\phi &= m\ddot{r} \\ \Rightarrow \ddot{r} + g\sin\phi &= 0 \\ F_\phi = \dot{p}_\phi \Rightarrow -mgr\cos\phi &= m\frac{\mathrm{d}}{\mathrm{d}t} \\ \Rightarrow r^2\ddot{\phi} + 2r\dot{r}\dot{\phi} + gr\cos\phi &= 0 \end{aligned}$$

We'll stop here with this example. Clearly, picking coordinates wisely is critical!

Note: you can use $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ *and compute* $\dot{x}(q, \dot{q})$*, ...*

In our example, this would be

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m\left(\left(\frac{d}{dt}(r\cos\phi)\right)^2 + \left(\frac{d}{dt}(r\sin\phi)\right)^2\right)$$

Here is the proof:

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$x = x(q) \Rightarrow \dot{x} = \sum \frac{\partial x}{\partial q_i}\dot{q}_i$$

$$\Rightarrow \dot{x}^2 = \left(\sum_j \frac{\partial x}{\partial q_j}\dot{q}_j\right) \times \left(\sum_k \frac{\partial x}{\partial q_k}\dot{q}_k\right)$$

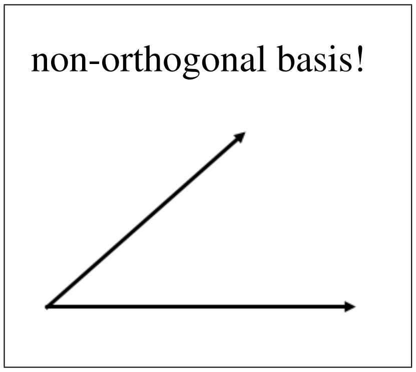
$$= \sum_{jk} \frac{\partial x}{\partial q_j}\frac{\partial x}{\partial q_k}\dot{q}_j\dot{q}_k$$

now sum over x, y, z to get all terms with $\dot{q}_j\dot{q}_k$

$$\Rightarrow T = \frac{1}{2}m\sum_{jk} a_{jk}\dot{q}_j\dot{q}_k$$

$$a_{jk} = \sum_{i \in x,y,z} \frac{\partial r_i}{\partial q_j}\frac{\partial r_i}{\partial q_k}$$

When do we get terms with $a_{jk} \neq 0$ for j $\neq$ k? (“off-diagonal” terms)



$$q_1 = x, q_2 = x + y$$

$$\Rightarrow a_{12} = \frac{\partial x}{\partial q_1}\frac{\partial x}{\partial q_2} + \frac{\partial y}{\partial q_1}\frac{\partial y}{\partial q_2} = 1 + 0 = 1$$

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