Massachusetts Institute of Technology

8.223, Classical Mechanics II

Exercises 1

1. Consider the motion of an object close to the surface of the Earth moving under the influence of Earth's gravity.

   a) The gravitational force law
   $$\vec{F} = -\frac{GMm}{r^2}\hat{r}$$
   reduces to
   $$\vec{F} = -gm\hat{z}$$
   at the surface with z pointing away from the center of the Earth (i.e., "up"). Express g in terms of G, $M_\oplus$ and $R_\oplus$. Compute g in SI units.

   b) For a cannonball fired at velocity v_o and angle θ above the horizon **on the Moon**, find the range; the distance from the cannon at which the cannonball hits the surface of the Moon. (You will need to recompute g for the Moon, but the lack of air on the Moon makes this calculation much easier there than on Earth.)

2. Hooke's law for a spring of constant k is $F = -kx$. A mass m is pushed from position x_o with velocity v_o at $t = 0$. Find the subsequent motion, $x(t)$.

3. Some exercises with coordinate systems

   a) Find the conversion from spherical (aka "polar") to cartesian coordinates, i.e. the functions $x(r, \theta, \phi)$, $y(r, \theta, \phi)$ and $z(r, \theta, \phi)$. Also find the inverse functions, $r(x, y, z)$, etc.

   b) Find the conversion from cylindrical to cartesian coordinates, i.e. the functions $x(\rho, \phi, z)$, etc. Also find the inverse functions, $\rho(x, y, z)$, etc.

   c) A particle starting from the origin moves in the $\hat{r}$ direction with velocity v_o and polar angles ϕ_0 and θ_0, such that $\vec{v} = v_o\hat{r}$. Express $\vec{v}$ in cartesian coordinates.

   d) OPTIONAL: A particle starting from a point $\vec{x}_o$ moves with the same velocity from part c) above (now no longer in the $\hat{r}$ direction). Express $\vec{v}$ in polar coordinates. From this exercise, you should notice a BIG weakness of the polar coordinate system.

4. Consider the equation of motion for a pendulum swinging under the influence of Earth's gravity

$$ml^2\ddot{\phi} = -mgl\sin\phi$$

a) Derive this equation of motion (using the Euler-Lagrange equation), with $\phi = 0$ corresponding to the y-axis (parallel to gravity). Recall that we implement the constraint of a rigid pendulum rod as $x = \ell\sin(\phi)$, $y = \ell\cos(\phi)$.

b) Justify that for small oscillations, we may use $\sin\phi \sim \phi$.

c) Solve the resulting equation for arbitrary (but small) initial conditions to give $\phi(t)$.

5. Starting from the Principle of Least Action, show that a system with N degrees of freedom and the Lagrangian $L(q_1, \cdots, q_N, \dot{q}_1, \cdots, q_N)$, gives N Euler-Lagrange equations $\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$ for all $i \in \{1, \cdots, N\}$.

6. Referring to Eq. 2.8 and the integral equation which follows in Landau,

a) explain in words why S and S' "differ by a quantity which gives zero on variation".

b) give 2 example functions which can be written as a total time derivative. That is, find 2 examples of $g(t)$ such that

$$g(q, \dot{q}, t) = \frac{d}{dt}f(q, t)$$

c) give 2 example functions which **cannot** be written as a total time derivative (i.e., find 2 examples of $g(q, \dot{q}, t)$ such that no $f(q, t)$ can be found for the equation in part b).

(Please make your 2 example functions for parts b and c at least linearly independent!)

7. For a free particle, we know that $L = L(v^2)$ if the particle moves in some direction with velocity v. Carry out a Galilean transformation to a primed frame moving with velocity $\vec{u}$ with respect to the original (unprimed) frame

$$\vec{x}' = \vec{x} - \vec{u}\,t, \quad t' = t$$

and show that if the unprimed frame is inertial, the primed frame is as well. Hint: you can take the explicit form of the Lagrangian for a free particle $L = \frac{1}{2}mv^2$, and show that L in the unprimed frame and L in the primed frame (i.e. $L' = L((\vec{u} - \vec{v})^2)$) give the same equations of motion.

8. Non-commuting derivatives: Use the definition of the total time derivative to

a) show that

$$\frac{\partial}{\partial q}\dot{f} = \frac{d}{dt}\frac{\partial}{\partial q}f$$

i.e., these derivatives commute for any function $f = f(q, \dot{q}, t)$.

b) show that

$$\frac{\partial}{\partial \dot{q}}\dot{f} = \frac{d}{dt}\frac{\partial}{\partial \dot{q}}f + \frac{\partial}{\partial q}f$$

(i.e., these derivatives do NOT commute.)

9. Take $L = \frac{1}{2}mv^2 - mgz$

a) Find the equations of motion.

b) Take $\vec{x}(0) = 0, \vec{v}(0) = \vec{v}_0, v_{0z} > 0$ and find $\vec{x}(\tau)$ and $\vec{v}(\tau)$, τ such that $z(\tau) = 0, \tau \neq 0$

10. A particle of mass m is confined to a parabolic surface of rotation $z = a\rho^2$, where $\rho = \sqrt{x^2 + y^2}$. The gravitational potential is $U = mgz$.

a) Show the Lagrangian is

$$L = \frac{1}{2}m(\dot{z}^2 + \dot{\rho}^2 + \rho^2\dot{\phi}^2) - mgz$$

subject to the constraint that $z = a\rho^2$, i.e. the particle remains on the surface of parabolic rotation.

b) Use the constraint equation to eliminate z from the Lagrangian and give the equations of motion.

c) Find the generalized momenta p_ρ and p_ϕ from the Lagrangian. Which, if any, are conserved and why?

d) OPTIONAL: Solve the equations of motion. Don't spend too much time on this!