8.13-14 Experimental Physics I & II "Junior Lab"

Fall 2007 - Spring 2008

Relativistic Dynamics: The Relations Among Energy, Momentum, and Velocity of Electrons and the Measurement of e/m

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This experiment is a study of the relations between energy, momentum and velocity of electrons moving at nearly the speed of light. Its goals are to compare the measured relations with the formulas of classical and relativistic dynamics, and to determine the value of e/m.

1. PREPARATORY QUESTIONS

1. Compare the classical and relativistic relations between energy, momentum, and velocity.

2. The source of high-energy electrons used in this experiment is the radioactive isotope ^{90}Sr and its decay product ^{90}Y. Describe the decay process of these isotopes and the energy spectra of the electrons (beta rays) they emit.

3. How does a semi-conductor diode detector work? Estimate the charge of the signal from a 280 keV electron in Si (0.5 eV per pair)

4. How good a vacuum is necessary for the experiment?

5. Compute the voltage required to cancel the magnetic force on the electrons in the velocity selector when the magnetic field is 110 Gauss. Assume the dimensions of the apparatus given in Figure 1, the book value of e/m, and the validity of relativistic mechanics. (The result will be a clue to where in the selector-voltage range you should search for the maxima in the counting rate for a typical magnetic field.)

6. A velocity selector is made of two parallel plates separated by a distance $d = 0.185$ cm. Using the same parameters of the previous problem, a) What is the spread in momentum of electrons that can enter the selector? b) What ΔV do you need to pass the highest and lowest momentum?

1.1. Units in this lab guide

This lab guide uses Gaussian units in which the force on a moving charge in a static field is

$$\vec{F} = q\vec{E} + \frac{q}{c}(\vec{v} \times \vec{B}).$$

The units are:
F—dyne
q—statcoulomb (esu)
B—gauss
E—statvolts
v,c—cm sec^{-1}

Some useful constants:

$$\begin{aligned} e &= 4.80298 \times 10^{-10} \text{ statcoulombs} \\ mc^2 &= 511 \text{ keV} = 8.18727 \times 10^{-7} \text{ ergs} \\ 1 \text{ volt} &= 3.336 \times 10^{-3} \text{ statvolts} \end{aligned}$$

1.2. Suggested Progress Check at end of 2nd Session

With the spectrometer magnetic field set to 100 gauss, plot the electron count rate at the detector versus velocity selector voltage (converted to units of β. Also, plot at least three data points of various kinematic energies versus β.

2. INTRODUCTION

Between 1900 and 1910 the relation between the energy, momentum and velocity of charged particles moving at high speeds was a central problem of physics. The fundamental contradictions between Newtonian mechanics and the Maxwell theory of the electromagnetic field, revealed most dramatically in the failures of the Michelson-Morley experiment to detect absolute motion of the earth through the "aether", barred the way to a logically consistent understanding of the deflection of a charged particle by electric and magnetic fields when the particle is moving at a velocity approaching the velocity of electromagnetic waves. Various formulas were derived by Abraham, Lorentz, and Poincare. In 1901 Kaufmann, using the new vacuum techniques pioneered by Thomson, determined the "apparent mass of the electron" by measuring the deflections of the recently discovered β rays from radioactive substances. There was considerable confusion as to whether the experimental results confirmed or contradicted one or another of the formulas. Then Albert Einstein, at the time an obscure 25-year-old examiner in the Swiss patent office, provided a clear and compelling theory of "the electrodynamics of moving bodies" which came to be called the special theory of relativity. Among its remarkable predictions was the slowing of clocks moving at high speed (demonstrated in the Junior Lab experiment on muon decay), and the non-classical relations between momentum, energy and velocity that are demonstrated in the present experiment [1].

In classical dynamics a particle acted on by a force $\vec{F}$ for a time dt over a displacement $\vec{dr}$ gains momentum $\vec{dp} = \vec{F}dt$ and kinetic energy $dK = \vec{F} \cdot \vec{dr}$.

According to Newtonian dynamics the kinetic energy K, momentum $\vec{p}$, and velocity $\vec{v}$ of a particle are related by the equations

$$\vec{p} = m\vec{v} \tag{1}$$

and

$$K = p^2/2m \tag{2}$$

where $p^2 = \vec{p} \cdot \vec{p}$ and m is the inertial mass of the particle. In classical mechanics there is no limit on the magnitude of $\vec{v}$.

According to relativistic dynamics (see [2], or other text on special relativity), these quantities are related by the equations

$$\vec{p} = m\gamma\vec{v} \tag{3}$$

and

$$E^2 = p^2c^2 + m^2c^4, \tag{4}$$

where $E = K + mc^2 = \gamma mc^2$ is the "total relativistic energy." The quantity m is a relativistic invariant and identical with the classical inertial mass, and γ is the Lorentz factor defined by the equation

$$\gamma = (1 - \beta^2)^{-1/2}, \tag{5}$$

where $\beta = |\vec{v}|/c$. Solving equation (4) for the kinetic energy one obtains

$$K = mc^2 \left(\left[1 + \left(\frac{p}{mc} \right)^2 \right]^{1/2} - 1 \right). \tag{6}$$

In the limit of high velocities where $p \gg mc$, equation (6) approaches

$$K = pc, \tag{7}$$

which is the exact relation between the energy and momentum of massless particles such as photons and neutrinos. For $p < mc$ equation (6), expanded by the binomial theorem, becomes

$$K = \frac{p^2}{2m} \left[1 - \frac{1}{4} \left(\frac{p}{mc} \right)^2 + \cdots \right]. \tag{8}$$

In the limit of low velocities ($|\vec{v}| \ll c$) where $p \ll mc$, equation (8) reduces to the classical relation expressed by equation (2).

The electromagnetic force on a charged particle is

$$\vec{F} = q \left[\vec{E} + \left(\frac{\vec{v}}{c} \right) \times \vec{B} \right], \tag{9}$$

where q is the invariant charge, $\vec{E}$ and $\vec{B}$ are the electric and magnetic field intensities, respectively, and c is the invariant speed of light. This force law is valid in both classical and relativistic dynamics. In this experiment you will measure the effects of electromagnetic force on the motion of electrons with velocities up to about $0.8c$, and measure directly the relations between $\vec{v}$, $\vec{p}$, and K.

Some useful references related to beta-decay, beta-spectroscopy and this experiment in particular are given in [3–8].

3. EXPERIMENTAL SETUP

The apparatus for the present experiment, shown schematically in Figure 1, is contained in a vacuum chamber inside a spherical magnet that maintains a uniform field in the vertical direction. Inside the vacuum chamber are 1) a source of energetic electrons (a minute quantity of ^{90}Sr which emits electrons with a spectrum of energies up to 0.546 MeV yielding a decay product ^{90}Y which emits electrons with a spectrum of energies up to 2.27 MeV), 2) baffles that reduce background counts due to scattered electrons that bounce around the vacuum chamber, 3) a narrow slit in the baffles at the 90-degree position around a circular path from the source, 4) a narrow slit that defines the radius of curvature of the electrons that enter the velocity selector, and 5) a solid state PIN diode detector. An electron, emitted with a momentum in a narrow range of magnitude and direction, traverses a helical path of fixed radius in the magnetic field and enters the gap between the velocity-selector plates. If the voltage, V, between the plates is adjusted so that the electric force cancels the magnetic force, then the electron passes through the "velocity selector" in a nearly straight trajectory and strikes the PIN diode in which a number of silicon valence electrons proportional to the energy deposited are promoted to the conduction band of the semiconductor and collected as a pulse of charge. The latter is converted by a low-noise preamplifier and precision amplifier into a voltage pulse with an amplitude proportional to the deposited energy which is measured by a multi-channel pulse height analyzer (MCA).

For each of several settings of the magnet current one measures 1) the magnetic field, 2) the voltage across the selector plates that yields the maximum counting rate of pulses in the narrow range of pulse height channels of the MCA corresponding to the energy range of the detected electrons, and 3) the median channel of the pulse height distribution. Given these data, the dimensions of the apparatus, and the energy calibration of the PIN detector one can determine the momentum, energy, and velocity of electrons and estimate the values of e/m, m, and e.

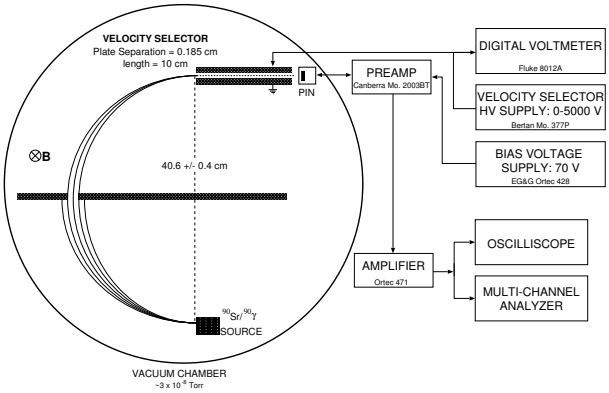

FIG. 1: Schematic diagram of the particle spectrometer and associated circuitry. The velocity selector is labeled VS, and the diode detector PIN.

4. THEORY OF THE EXPERIMENT

In the region between the source and the velocity selector, where only a magnetic field exists, the motion is described by the equation

$$\frac{e|\vec{v}|B}{c} = \left|\frac{d\vec{p}}{dt}\right| = \omega|\vec{p}| = \frac{|\vec{v}|p}{\rho}, \tag{10}$$

so

$$B = \left(\frac{c}{e\rho}\right) p, \tag{11}$$

where ρ is the radius of curvature of the particle trajectory under the influence of the magnetic force. Placement of the source, the collimator, and the aperture of the velocity selector on a circle of radius ρ allows only particles with a momentum in a narrow range around $Be\rho/c$ to enter the velocity selector. In the region between the velocity selector plates $\vec{E}$, $\vec{B}$, and $\vec{v}$ are perpendicular to each other so one can write for particles that experience zero deflecting force and go exactly parallel to the plates the relation

$$eE - \frac{evB}{c} = 0, \tag{12}$$

where $E = V/d$. Hence

$$\beta = |\vec{v}|/c = E/B. \tag{13}$$

Thus, for any combination of E and B such that $E < B$, the velocity selector transmits particles with velocities near E/B in a narrow range of magnitudes whose width depends on the geometry of the gap between the plates.

A plot of measured values of B against the ratio E/B reveals the relation between momentum and velocity. According to the classical equation (1) the plot would be fit by a straight line with a slope of $(mc^2)/(e\rho)$. Deviation from a straight line as $E/B \to 1$ indicates the failure of the classical relation between momentum and velocity as the velocity approaches c. According to the relativity equation (3), a plot of B against $(E/B)[1-(E/B)^2]^{-1/2}$ should be fit by a straight line with a slope of $(mc^2)/(e\rho)$. From the slope and knowledge of the values of c and ρ one can estimate the invariant quantity e/m.

In the experiment it is a good idea to set the magnetic field and then determine the voltage between the selector plates that gives the highest rate of counts of electrons that traverse the circular path and pass between the velocity selector plates to strike the PIN diode detector.

Note that measurements of E and B alone yield a determination of e/m but neither e nor m separately. This is characteristic of all experiments involving only electromagnetic forces. Why is this so? Consider the analogy to the problem of determining the ratio of gravitational to inertial mass of a body moving under the influence of gravity. (Incidentally, there are two experiments in Junior Lab which yield measurements of e, the Millikan oil drop experiment, and the shot noise experiment.)

The PIN diode detector combines the virtues of an ultra-thin entrance window and surface dead layer with a total sensitive thickness sufficient to stop electrons with several hundred keV of kinetic energy. Thus, with appropriate calibration, the PIN diode provides a measure of the kinetic energy of the detected electrons. Plots of the kinetic energy against E/B or against $[1-(E/B)^2]^{-1/2}$ reveal the relation between energy and velocity; the slope of the latter plot yields a value of m (or more conveniently mc^2 expressed in units of keV in terms of which the energies of the calibration X and gamma-ray photons are expressed). A plot of energy against B reveals the deviation of the energy-momentum relation from the classical quadratic form $E = p^2/(2m)$ toward the linear form $E = pc$ valid for a particle moving with a velocity close to c.

5. APPARATUS DETAILS

The magnet consists of a stack of circular air-core coils enclosing a spherical volume and connected in series so as to produce a current distribution over the surface of the sphere which is approximately equal to the ideal smooth distribution required to produce a uniform field inside the sphere. It turns out that the required distribution of surface current density is

$$J(\theta, \phi) = J_0 \sin\theta, \tag{14}$$

where J_0 is related to the magnitude $B_i = |\vec{B}_i|$ of the uniform field inside by an equation that is left for the reader to derive (for hints see Appendix A).

The current for the magnet is provided by a stabilized supply whose output can be as high as 200 volts at low impedance, which means that it can deliver a lethal shock. **Be careful! Turn off the power supply before touching the magnet. Be sure the cooling fan is turned on whenever the current exceeds 3 amperes.**

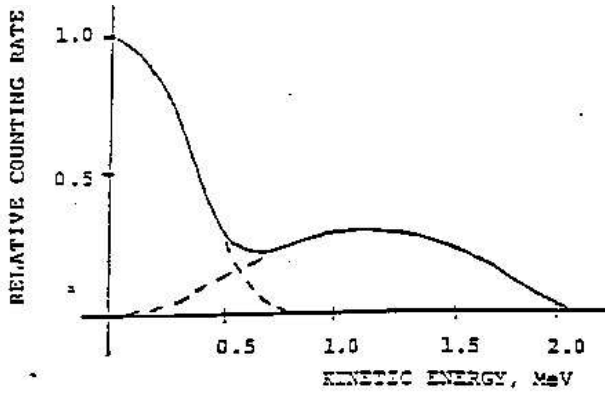

FIG. 2: The β-ray Spectra of ^{90}Sr and ^{90}Y. The actual energy spectrum of the electrons in this experiment will be distorted by energy loss of the electrons in the material of the source and in the thin metal covering over the source.

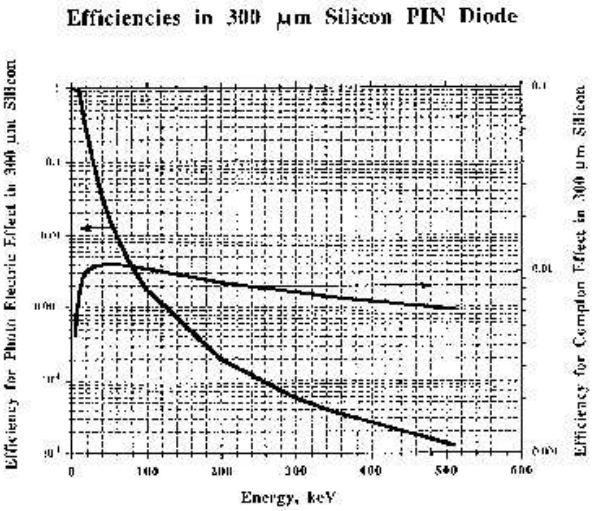

FIG. 3: Photon detection efficiencies in silicon.

The coil assembly is split at its equator. The upper hemisphere can be lifted with the block and tackle for access to the spectrometer assembly which is located inside the vacuum vessel under a heavy plastic plate.

The β-ray source, ^{90}Sr, decays with a half life of 28 yrs to ^{90}Y with emission of an electron with a maximum energy of 0.546 MeV. The yttrium nucleus in turn decays with a half life of 64 hrs to ^{90}Zr with emission of an electron with a maximum energy of 2.27 MeV. In each of these steps, a neutron in the nucleus is transformed into a proton with the simultaneous emission of an electron and an antineutrino. Since the two particles emitted in the β-decay process can share the available energy with the recoiling nucleus in any proportion consistent with the conservation of momentum, the energy spectra of the electrons are continuous from zero to the specified maxima (for details see [4]). A sketch of the energy spectrum of electrons emitted by a thin ^{90}Sr source is displayed in Figure 2.

A narrow slit located at the 90-degree position defines the circular trajectory. The velocity selector consists of two aluminum blocks mounted with their faces precisely parallel to each other and tangent to the central electron path at the entrance to the gap. The voltage for the velocity selector is provided by a high-impedance high-voltage power supply supplemented by a high-voltage battery.

The detector is a PIN diode shielded from light by a thin film of aluminized Mylar. The PIN diode should be operated with a positive bias of 50 volts (**caution: voltage above 70 volts will destroy it**). A graph of various interaction efficiences for photons in silicon is shown in Figure 3.

6. EXPERIMENT

Check that the vacuum is better than 10^{-4} torr. A good vacuum is essential because multiple small angle scattering of the electrons by residual gas will distort the trajectories. The rms angle of scattering can be estimated by the following approximate formula:

$$<\theta^2>^{1/2} = \frac{15 \text{ MeV}/c}{p}\left(\frac{X}{X_0}\right)^{1/2}, \qquad (15)$$

where p is the momentum measured in MeV/c (1 MeV/c $= 5.34 \times 10^{-17}$ dyne-sec), X is the amount of material in the path measured in gm cm^{-2}, and X_0 is a constant, called the "radiation length" and characteristic of the material. The radiation length of air is 37 gm cm^{-2}.

Set up the pulse height measurement chain, and check the performance at each stage with the oscilloscope.

The magnetic field is measured with a Hall effect magnetometer. **Before every use of the magnetometer check the calibration and the zero setting.** The Hall sensor, fragile and expensive, is a thin wafer of semiconductor attached to the signal leads. The wafer and its attached wires should be carefully inserted in the protective aluminum tube and the tube inserted in the plastic block that can be placed on top of the plastic cover of the vacuum chamber. To measure accurately the vertical component of the magnetic field the plane of the semiconductor wafer must be exactly horizontal. To achieve this condition raise the top hemisphere of the magnet just enough to allow you to place the plastic block with the Hall sensor flat inside the aluminum tube in the middle of the vacuum cover where you can assume the field will be almost exactly vertical. Turn on the magnet current to 5 amperes. Rock the plastic block to find the orientation that yields the maximum magnitude of the field. Then rotate the aluminum tube containing the Hall sensor so that the maximum reading is obtained with

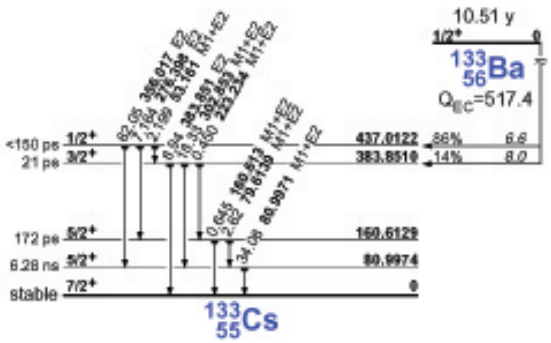

FIG. 4: Decay of Ba-133 to Cs-133. Data from the Table of Nuclides.

the plastic block lying flat on the vacuum cover.

Explore the operation of the experiment, and survey the ranges of the measurements before starting a definitive sequence of measurements:

1. With the magnet current set to give a field of approximately 100 gauss, measure the precise values of the field at several positions above the circular trajectory from source to PIN diode with the magnet closed to ascertain the degree of uniformity of the field and estimate error in the field measurement due to non-uniformity.

2. Calibrate the measurement chain with the X-ray and gamma-ray emissions of the laboratory calibration sources. Be aware of the possibility of a zero offset, i.e., of the 0 channel corresponding to some positive energy. Specially useful for the calibration is the ^{133}Ba source stored in the lead container under the spherical magnet (^{133}Ba decays by electron capture, so in addition to the gamma rays of several energies emitted by the daughter nucleus, the daughter atom, Cs, also emits a K X-ray). You can find the gamma-ray energies by examination of Figure 4 taken from `http://ie.lbl.gov/toi.html`. The K-shell X-ray energies can be found in various X-ray tables (e.g. CRC handbook). Handbook.

   The cross section for photoelectric interaction of high-energy photons of energy E in matter of atomic number Z varies approximately as ZE. In the energy range from ~ 20 to 100 keV monoenergetic photons interacting in the silicon PIN diode produce a pulse-height spectrum with a prominent photoelectric peak. Above 200 keV the photoelectric interaction cross section is small relative to the Compton scattering cross section. However, you can use the high end (Compton edge) of the distribution of pulse heights produced by recoil electrons from Compton scattering of the photons as a calibration point. In the text of Melissinos you can find a discussion of Compton scattering, and the relevant information for calculating the energy corresponding to the Compton edge of the distribution of pulse heights produced by electrons that recoil from the scattering events. **The first day you should acquire a 2-3 hour (or overnight if possible) calibration in order to generate sufficient events at higher energies for a multi-point calibration. Plot as many calibration energies as possible against the median MCA channels of the corresponding spectrum features (photopeak or Compton edge). The resultant calibration should be very linear if you've correctly identified all the features. On subsequent days, a 10-15 minute calibration should suffice at the start of each lab session.**

3. With the magnet field at approximately 100 gauss, vary the voltage applied to the velocity selector around 4.5 kV while looking for evidence of pulses accumulating in a narrow distribution in the spectrum of pulse heights displayed on the MCA screen. Increase the energy of the selected electrons step by step by increasing B and V, and adjust the measurement chain so that the pulses recorded with the highest attainable energies accumulate near the high end of the spectrum displayed on the MCA.

4. For each of many (~ 10) values of B spread over as wide a range as possible, measure the voltage V that yields the maximum counting rate. The best procedure is to measure the counting rate as a function of V around V and at enough values of V to permit a gaussian fit to the data. Tabulate B, V, and the kinetic energy corresponding to the median channel number of the pulse-height distribution. Estimate the errors in each of these measurements so that you have the necessary data for error analysis of your results. (Note: the error of a voltage determination is not the FWHM of the measured curve of V vs count rate.)

7. ANALYSIS

For each value of B and $E = V_0/d$ calculate E/B. Then plot E/B on the Y axis against B on the X axis. On the same plot draw curves that show the relations between these quantities predicted by classical and relativity dynamics.

For each value of B, calculate e/m according to the classical and the relativity formulas and plot the results with error bars against B.

Discuss the effect of the spread in the trajectories of the detected electrons on the errors in the determinations of e/m.

Derive a final best value of e/m with a probable error using the techniques in [9].

Plot your measured values of K against p, and draw the curves predicted by classical and relativity mechanics.

7.1. Possible Theoretical Topics

1. The surface current distribution required to produce a uniform field inside the spherical magnet.
2. Relativistic dynamics—-i.e., the relation between v, p, and E.
3. Compton scattering and the photoelectric effect at gamma-ray energies.
4. PIN diodes as particle detectors.
5. Resolution of the velocity selector.
6. Multiple Coulomb scattering of the electrons by the residual air in the vacuum chamber.

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[1] Various, selected reprints by the American Institute of Physics (1963).
[2] A. P. French, *Special Relativity* (MIT Press, 1968).
[3] E. Bleuler and G. Goldsmith, *Experimental Nucleonics* (Holt, Rinehart, and Winston, 1960), pp. 342–346.
[4] R. Evans, *The Atomic Nucleus* (McGraw-Hill, 1953), chapter 17, pp. 536–566, Also contains statistics in chapter 26.
[5] J. England, *Techniques in Nuclear Structure Physics*, vol. Vol. II (Wiley and Sons, 1974), pp. 349 ff.
[6] R. B. L. R. P. Feynman and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, 1963), i-15.
[7] J. D. Jackson, *Classical Electrodynamics* (Wiley and Sons, 1962).
[8] D. J. Griffiths, *Introduction to Electrodynamics* (Upper Saddler River, 1999).
[9] P. R. Bevington and D. K. Robinson, *Data Reduction and Error Analysis for the Physical Sciences* (McGraw-Hill, 2003), 3rd ed.

APPENDIX A: THE SPHERICAL UNIFORM-FIELD MAGNET

The magnet is an arrangement of currents in coaxial coils distributed over the surface of a sphere in such a way as to produce a uniform magnetic field B_i inside the sphere. The design problem is to find the distribution of surface current density $J(\theta, \phi)$ that achieves this condition.

Everywhere but on the surface of the sphere curl $B = j = 0$. It follows that $B = -\text{grad}\ \phi_m$, where ϕ_m is a (magnetic) potential function, i.e. a solution of Laplace's equation div (grad ϕ_m) $= \nabla^2 \phi_m = 0$. According to a fundamental theorem of potential theory, a solution of Laplace's equation that satisfies a particular set of boundary conditions, such as the distribution of currents on the surface of our sphere, is unique except for an additive constant. Thus if you can cook up a solution of Laplace's equation that does, indeed, satisfy the boundary conditions, you can be sure you have got the one and only solution to the problem.

If the field is to be uniform inside the sphere, then

$$\phi_m(r, \theta, \phi) = -B_i z = -B_i r \cos\theta. \qquad r < R \qquad \text{(A1)}$$

Outside at large distances it is certainly true that

$$\phi_m = \frac{m \cos\theta}{r^2} \qquad r \gg R \qquad \text{(A2)}$$

i.e. the potential must have the form of a dipole potential at large distances since the field is the sum of contributions from many coaxial circular current loops. The problem boils down to that of matching an outside dipole field to the inside uniform field B_i by proper adjustment of the surface current distribution in such a way as to satisfy the continuity requirements on the radial and tangential components of $\vec{B}$. The continuity conditions follow directly from the two Maxwell equations governing the magnetostatic field, *viz.*

$$\begin{aligned} \nabla \cdot \vec{B} &= 0, \\ \nabla \times \vec{B} &= \left(\frac{4\pi}{c}\right) \vec{j}. \end{aligned}$$

(Note: $\vec{j}$ is the volume current density with dimensions of statamps cm^{-2} , whereas what we seek is the surface current density with dimensions statamps cm^{-1}.)

The rest of the solution is left to the reader.

For the general approach to the solution of such boundary condition problems see, e.g., Jackson (1962) or Griffiths (1999).

APPENDIX B: EQUIPMENT LIST

Manufacturer	**Description**	**URL**
Agilent	Oscilloscope	agilent.com
Sorenson	0-5A Power Supply	
RFL	Gaussmeter	
Bertran	7.5kV Velocity Selector PS	
Canberra 2006	Charged Particle Preamp	
Hamamatsu	PIN Diode	
Ortec	Amplifier	ortec-online.com