8.04: Quantum Mechanics | Professor Allan Adams
Massachusetts Institute of Technology | Tuesday March 5

Problem Set 5

Due[†] *Tuesday March 12 at 11.00AM*

Assigned Reading:

E&R	6_9, App-I
Li.	7_{1-4}
Ga.	4_7, $6_{1,2}$
Sh	6_{all}, 7_{all}

---

1. **(10 points) The Probability Current**

   A particle is in a state described by the wavefunction $\psi(x,t)$. Let $P_{ab}(t)$ be the probability of finding the particle in the range $a < x < b$ at time t. Show that

   $$\frac{dP_{ab}}{dt} = \mathcal{J}(a,t) - \mathcal{J}(b,t) \ ,$$

   where

   $$\mathcal{J}(x,t) = \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)$$

   What are the units of $\mathcal{J}(x,t)$?

   Hint: What is $P_{ab}(t)$ in terms of $\psi(x,t)$? Remember that time-derivatives of $\psi(x,t)$ are determined by the Schrödinger equation. Use integration by parts as often as you need.

   Aside: $\mathcal{J}(x,t)$ is called the probability current because it tells you the rate at which probability is flowing past the point x. If $P_{ab}(t)$ is increasing, then more probability is flowing into the region at one end than flows out at the other; $\mathcal{J}$ measures the magnitude of this current.

2. **(20 points) Visual observation of a quantum harmonic oscillator**

   An experimenter asks for funds to observe visually through a microscope the quantum behavior of a small oscillator. According to his proposal, the oscillator consists of an object 10^{-4}cm in diameter with an estimated mass of 10^{-12}g. It vibrates on the end of a thin fiber with a maximum amplitude of 10^{-3}cm and frequency 10^3Hz. You are referee for the proposal.

   (a) What is the approximate quantum number for the system in the state described?

   (b) What would be its energy in eV if it were in its lowest-energy state? Compare with the average thermal energy (25meV) of air molecules at room temperature.

   (c) What would be its classical amplitude of vibration if it were in its lowest-energy state? Compare this with the wavelength of visible light (500 nm) by which it is presumably observed.

   (d) Would you, as referee of this proposal, recommend award of a grant to carry out this research?

3. **(30 points) Harmonic Oscillators Oscillate Harmonically**

A particle of mass m in a harmonic oscillator potential

$$V(x) = \frac{m\omega_0^2}{2}x^2$$

has an initial wave function

$$\psi(x, 0) = \frac{1}{\sqrt{2}}\left[\phi_0(x) + i\phi_1(x)\right]$$

where ϕ_0 and ϕ_1 are the n=0 and n=1 normalized eigenstates for the harmonic oscillator.

(a) Write down $\psi(x, t)$ and $|\psi(x, t)|^2$. (For this part, you may leave the expression in terms of ϕ_0 and ϕ_1.)

(b) Find the expectation value $\langle x \rangle$ as a function of time t. Notice that it oscillates with time. What is the amplitude of the oscillation (in terms of m, ω_0 and fundamental constants)? What is its angular frequency?

(c) Find the expectation value $\langle p \rangle$ as a function of time.

(d) Show that the probability distribution of a particle in a harmonic oscillator potential returns to its original shape after the classical period $T = 2\pi/\omega_o$. You should prove this for *any* harmonic oscillator state, including *non*-stationary states. What feature of the harmonic oscillator makes this true?

4. **(30 points) Operators for the Harmonic Oscillator**

   Let ϕ_0 be the normalized ground state of the harmonic oscillator, ie $\hat{a}\,\phi_0$=0 and $(\phi_0|\phi_0)$=1, where $\hat{a}$ is the annihilation (aka lowering) operator and $\hat{a}^\dagger$ is the creation (aka raising) operator. Recall the commutator $\left[\hat{a}, \hat{a}^\dagger\right] = 1$.

   (a) Show that the norm of the unnormalized energy eigenstate $\tilde{\phi_n} \equiv (\hat{a}^\dagger)^n \phi_0$ is $\sqrt{n!}$.

   (b) As a consequence, the normalized energy eigenstates take the form, $\phi_n \equiv \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n \phi_0$. Verify that these eigenstates are *orthonormal*, ie show that $(\phi_n|\phi_m) = \delta_{nm}$.

   (c) Show that $\hat{a}\phi_n = \sqrt{n}\phi_{n-1}$, and that $\hat{a}^\dagger \phi_n = \sqrt{n+1}\phi_{n+1}$.

   (d) Define the "number" operator $\hat{N} = \hat{a}^\dagger \hat{a}$. Show that $\left[\hat{N}, \hat{a}\right] = -\hat{a}$ and $\left[\hat{N}, \hat{a}^\dagger\right] = +\hat{a}^\dagger$. What are the eigenfunctions of $\hat{N}$, and what are its eigenvalues?

   (e) Express $\hat{x}$ and $\hat{p}$ in terms of $\hat{a}$ and $\hat{a}^\dagger$. Use these expressions to show that anytime the HO is in an energy eigenstate $\psi = \phi_n$, $\langle x \rangle$=0 and $\langle p \rangle$=0.

   (f) Express $\hat{x}^2$ and $\hat{p}^2$ in terms of $\hat{a}$ and $\hat{a}^\dagger$. Use these expressions to show that when the HO is in an energy eigenstate $\psi = \phi_n$,

$$\Delta x \Delta p = \frac{\hbar}{2}(2n+1)$$

Aside: The Meaning of $\hat{a}$ and $\hat{a}^\dagger$

We first met the creation ($\hat{a}^\dagger$) and annihilation ($\hat{a}$) operators while quantizing the harmonic oscillator, where they appeared as convenient complex combinations of the position and momentum operators,

$$\hat{a} = \frac{\hat{x}}{x_o} + i\frac{\hat{p}}{p_o}\ , \qquad \hat{a}^\dagger = \frac{\hat{x}}{x_o} - i\frac{\hat{p}}{p_o}\ .$$

At this point, given the connection between classical observables and quantum operators, it's natural to wonder, "what is the classical observable associated to $\hat{a}$?"

The short answer (longer answer below) is that there *isn't* one. In our postulates, *classical observables* correspond to operators with *real eigenvalues* (aka *Hermitian* operators, aka operators which are *self-adjoint*, $\hat{A}^\dagger = \hat{A}$, as discussed in the lecture notes). But by construction, $\hat{a}$ is *not* a Hermitian operator – due to the i in its definition, its adjoint is $\hat{a}^\dagger$ – and so does *not* in general have real eigenvalues. This means we cannot consider it as the quantum operator associated to any classical observable.

This might seem like a subtle point – it's tempting to say, look, i can always build a classical observable called **awesomeness** which is given by $\mathbf{a} = x + iy$, and since i can certainly measure both x and y, i can also measure $\mathbf{a}$. What's the problem?

The longer answer goes like this. In quantum mechanics, predicting the measurement of some observable $\hat{A}$ involves taking the wavefunction ψ, expressing it as a superposition of orthonormalized eigenfunctions of $\hat{A}$, and interpreting the coefficients of each eigenfunction as the probability amplitude to measure the associated eigenvalue:

$$\psi = \sum c_n\,\phi_n \qquad \hat{A}\phi_n = A_n\phi_n \qquad \mathbb{P}(A_n) = |c_n|^2\ .$$

Here's the crucial observation: this procedure assumes that it is *possible* to express an arbitrary wavefunction ψ as a superposition of eigenfunctions of $\hat{A}$, leading to a prediction for probabilities, and that this decomposition is sufficiently *unique* that it gives unambiguous predictions for $\mathbb{P}(A_n) = |c_n|^2$. This is an enormously non-trivial assumption which, for a generic operator with complex eigenvalues, is *spectacularly false!* The reason we focus on Hermitian operators is that we *can always* do this when our operator is Hermitian[1]. In mathematics, this is known as the the spectral theorem. Physically, this is to say that *anything we can measure* should map to a good operator. So that's the reason we harp on real observables and Hermitian operators – not because we can't intelligently discuss complex classical observables, but because *quantum measurement* doesn't make sense for generic operators with complex eigenvalues.

---

[1]More precisely, every Hermitian operator admits an orthonormal basis of eigenfunctions which is unique up to unitary equivalence – *e.g.* up to multiplying the c_n with phases which leave $\mathbb{P}(A_n) = |c_n|^2$ alone. Incidentally, requiring that $\mathbb{P}(A_n)$ is invariant under unitary equivalence explains why we pick $|c_n|^2$ and not $|c_n|^4$ or some other power – to see this, think about rotations between degenerate eigenfunctions.

This is *not* to say that non-hermitian operators are unphysical – indeed, $\hat{a}$ and $\hat{a}^\dagger$ are enormously illuminating operators which lie at the beating heart of quantum mechanics and field theory! We just can't directly associate them with classical observables.