8.022 FALL 2002 QUIZ#3 SOLUTIONS

Solution to Problem 1

$$Z_1 = \frac{3}{2i\omega C}$$

$$Z_2 = R + i\omega L$$

a) (5 pts)

$$Y_{tot} = \boxed{\frac{2i\omega C}{3} + \frac{1}{R+i\omega L}}$$

$$= \frac{R}{R^2+\omega^2L^2} + i\left[\frac{2\omega C}{3} - \frac{\omega L}{R^2+\omega^2L^2}\right]$$

$$\frac{2C}{3} = \frac{L}{R^2+\omega^2L^2}$$

(5 pts) b)

$$\boxed{\omega = \sqrt{\frac{3}{2LC} - \frac{R^2}{L^2}}}$$

(5 pts) c) $\tilde{I}_1 = Y_1 \tilde{V}$ $\qquad \tilde{V} = -iV_0 e^{i\omega t}$

$$= \frac{2i\omega C}{3}\tilde{V}$$

$$= +\frac{2}{3}\omega C V_0$$

$$I_1(t) = \boxed{+\frac{2\omega C}{3} V_0 \cos(\omega t)} = I_{2C}(t) = I_C(t)$$

$$\tilde{I}_2 = Y_2 \tilde{V}$$

$$= \frac{1}{R+i\omega L}(-iV_0) = \frac{1}{\sqrt{R^2+\omega^2L^2}} e^{i\tan^{-1}\left(\frac{-\omega L}{R}\right)}(-iV_0)$$

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$$I_2(t) = \boxed{\frac{V_0}{\sqrt{R^2+(\omega L)^2}} \sin\left(\omega t + \tan^{-1}\left(\frac{-\omega L}{R}\right)\right)}$$

$$= I_R(t) = I_L(t)$$

d) (5 pts) $\tilde{I}_0 =$

$$= \left[\frac{R}{R^2+\omega^2L^2} + i\left(\frac{2i\omega C}{3} - \frac{\omega L}{R^2+\omega^2L^2}\right)\right]\tilde{V}$$

$$= I_0 \sin(\omega t + \phi)$$

$$I_0 = \sqrt{\left(\frac{R}{R^2+\omega^2L^2}\right)^2 + \left(\frac{2\omega C}{3} - \frac{\omega L}{R^2+\omega^2L^2}\right)^2}$$

$$\phi = \tan^{-1}\left[\frac{\left(\frac{2\omega C}{3} - \frac{\omega L}{R^2+\omega^2L^2}\right)}{\left(\frac{R}{R^2+\omega^2L^2}\right)}\right]$$

$$= \tan^{-1}\left[\frac{2\omega C(R^2+\omega^2L^2)}{3R} - \frac{\omega L}{R}\right]$$

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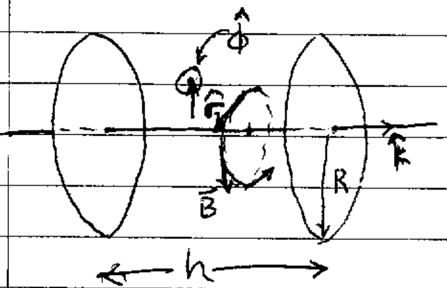
Problem #2: Displacement current



$$\vec{E} = E_0\left(1-\frac{r}{R}\right)\sin\omega t\,\hat{k}$$

(a) $\vec{J}_d = \frac{1}{4\pi}\frac{\partial \vec{E}}{\partial t} =$

$$\frac{E_0}{4\pi}\omega\left(1-\frac{r}{R}\right)\cos\omega t\,\hat{k}$$

(b). $\vec{J}_{conduction} = 0$. Symmetry $\vec{B} = B\hat{\phi}$

Ampere: $2\pi r B = \frac{4\pi}{c}\int_0^r J_d \cdot 2\pi r\,dr \Rightarrow$

$$B = \frac{4\pi}{cr}\cdot\frac{E_0\omega\cos\omega t}{4\pi}\int_0^r\left(1-\frac{r}{R}\right)r\,dr \Rightarrow$$

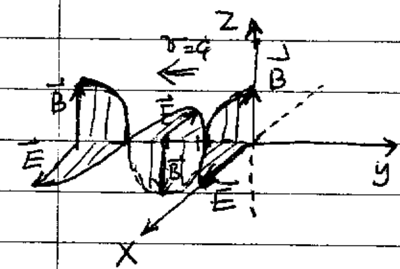
$$B = \frac{E_0\omega\cos\omega t}{cr}\left[\frac{r^2}{2}-\frac{r^3}{3R}\right] \Rightarrow$$

$$\vec{B} = \frac{E_0\omega\cos\omega t}{2c}\,r\,\underbrace{\left[1-\frac{2r}{3R}\right]}_{\text{NOTE } >0}\hat{\phi}$$

(c). B is azimuthal at $t=0$. $\cos\omega t = 1 \Rightarrow$

$$\vec{B} = \frac{E_0\omega}{2c}\,r\,\underbrace{\left(1-\frac{2r}{3R}\right)}_{>0}\hat{\phi}$$

Problem #3 Electromagnetic Wave



(a). $\vec{B} = B_0\cos(\omega t + ky)\,\hat{z}$
$\vec{E} = B_0\cos(\omega t + ky)\,\hat{x}$

(b) see adjacent figure

(c). $\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} = \frac{c}{4\pi}B_0^2\cos^2(\omega t + ky)(-\hat{y})$.

$$|P| = \left|\int\vec{S}\cdot d\vec{a}\right| = \frac{AcB^2}{4\pi}\cos^2(\omega t + ky).$$

Instantaneous

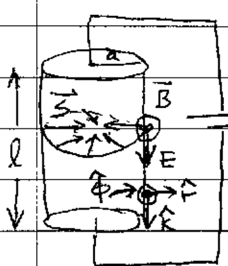
$$U = \int_0^{10T}|P|\,dt = \frac{AcB_0^2}{4\pi\omega}\underbrace{\int_0^{20\pi}\cos^2(\omega t + ky)\,d(\omega t)}_{=\frac{1}{2}\cdot 20\pi = 10\pi}.$$

$$\Rightarrow U = \frac{AcB_0^2}{4\pi\omega}\cdot 10\pi = \frac{5AB_0^2c}{2\omega}$$

$$[U] = \frac{[L]^2\cdot[U]\cdot[L]^{-3}\cdot[L]}{[T]\cdot\frac{1}{[T]}} = [U] \quad \text{TRUE!}$$

UNITS CONSISTENT!

Problem #4 : Poynting Vector



(a) $\vec{E} = \frac{V}{\ell}\hat{k}$

$\oint \vec{B}\cdot d\vec{\ell} = \frac{4\pi}{c} I \Rightarrow \vec{B} = \frac{2I}{ca}\hat{\phi}$

$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} = \frac{c}{4\pi}\frac{V}{\ell}\frac{2V}{caR}\underbrace{\hat{k}\times\hat{\phi}}_{-\hat{r}}$

$\vec{S}$ points inward

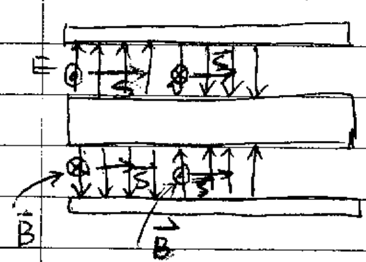
$= \frac{V^2}{2\pi a\ell\cdot R}(-\hat{r})$

(b) $\int_{\text{WALLS}} \vec{S}\cdot \overset{=da\,\hat{r}}{d\vec{a}} = -\frac{V^2}{R}\frac{1}{2\pi a\ell}\cdot 2\pi a\ell = -\frac{V^2}{R}$ ← OHMIC LOSSES

Energy "comes" from outside and expended inside

When polarity of Emf is inverted. Both $\vec{E}$ & $\vec{B}$ flip sign and thus $\vec{S}$ remains unchanged. Physically if $\vec{S}$ represents the "inward" flow of field energy transformed to ohmic losses, this should remain unchanged when polarity is changed as ohmic losses are independent of the direction of the current.

Problem #5 : Coaxial line



(a) show $\vec{B}, \vec{S}$: see figure

(b) $V = \int_a^b \vec{E}\, d\vec{r} =$

$= K\cos\left(\frac{\omega}{c}x - \omega t\right)\ln\frac{b}{a}$

$= V_{max}\cos\left(\frac{\omega}{c}x - \omega t\right)$

(c) Use Ampere's law to find I :

$\oint \vec{B}\cdot d\vec{\ell} = \frac{4\pi}{c} I \Rightarrow \frac{K}{r}\cos\left(\frac{\omega}{c}x - \omega t\right)\cdot 2\pi r = \frac{4\pi}{c} I \Rightarrow$

$I = \frac{cK}{2}\cos\left(\frac{\omega}{c}x - \omega t\right) = I_{max}\cos\left(\frac{\omega}{c}x - \omega t\right)$

(d) $\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} = \frac{c}{4\pi}\frac{K^2}{r^2}\cos^2\left(\frac{\omega}{c}x - \omega t\right)\hat{k}$

$\Rightarrow P = \int \vec{S}\, d\vec{a} = \frac{cK^2\cos^2\left(\frac{\omega}{c}x - \omega t\right)\hat{k}}{4\pi}\int_a^b \frac{1}{r^2}\, 2\pi r\, dr$

$= \frac{cK^2}{2}\underbrace{\cos^2\left(\frac{\omega}{x} - \omega t\right)}_{\langle\rangle \downarrow}\ln\frac{b}{a}$

$P_{ave} = \frac{cK^2}{2}\ln\frac{b}{a}\cdot\frac{1}{2} = \underbrace{\frac{cK}{2}}_{I_{max}}\cdot\underbrace{K\ln\frac{b}{a}}_{V_{max}}\cdot\frac{1}{2} = \frac{1}{2}I_{max}V_{max}$