8.012 Physics I: Classical Mechanics
Fall 2008

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.012 Fall 2007

Midterm Exam 1
Thursday, October 4, 2007

NAME: ______________ S O L U T I O N S ______________

Instructions:
1. Do all **FIVE (5)** problems. You have **90 minutes**.
2. **SHOW ALL WORK**. Be sure to **circle your final answer**.
3. Read the questions carefully.
4. All work must be done in this booklet. Extra blank pages are provided.
5. NO books, notes, calculators or computers are permitted. **A sheet of useful equations is provided on the last page**.

Your Scores

Problem	Maximum	Score	Grader
1	10		
2	20		
3	25		
4	20		
5	25		
Total	100		

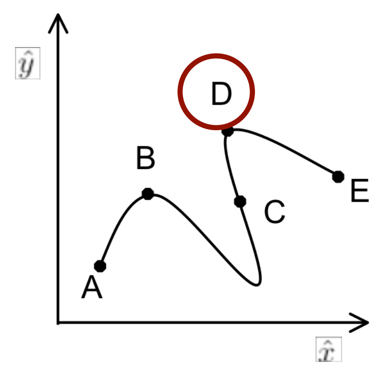
Problem 1: Quick Multiple Choice Questions [10 pts]

For each of the following questions circle the correct answer. You do not need to show any work.

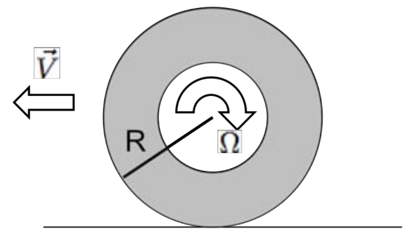
(a) A block with mass M and contact area A slides down an inclined plane with friction, covering a distance L in time T. How much time does it take another block with the same mass and composition, but twice the surface area, to slide down the same length?

0.5T T 2T T^2

(b) A particle moves with constant speed from point A to point E on the path shown to the right. Among the points B, C and D, circle the point where the magnitude of acceleration is greatest.

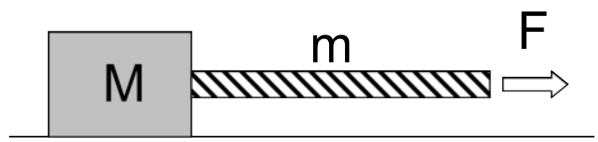


(c) A tire rolls on a flat surface with constant angular velocity Ω and velocity $\vec{V}$ as shown in the diagram to the right. If V = ΩR, then in which direction does friction from the road act on the tire?



Toward the left Toward the right Friction does not act on the tire

(d) You pull a brick of mass M sitting on a flat table using a thick rope of mass m (see picture to right). Friction between the surface of the table and the brick is sufficient to prevent the brick from moving. Compared to the force you pull one end of the rope, the force that the other end of the rope exerts on the brick is:

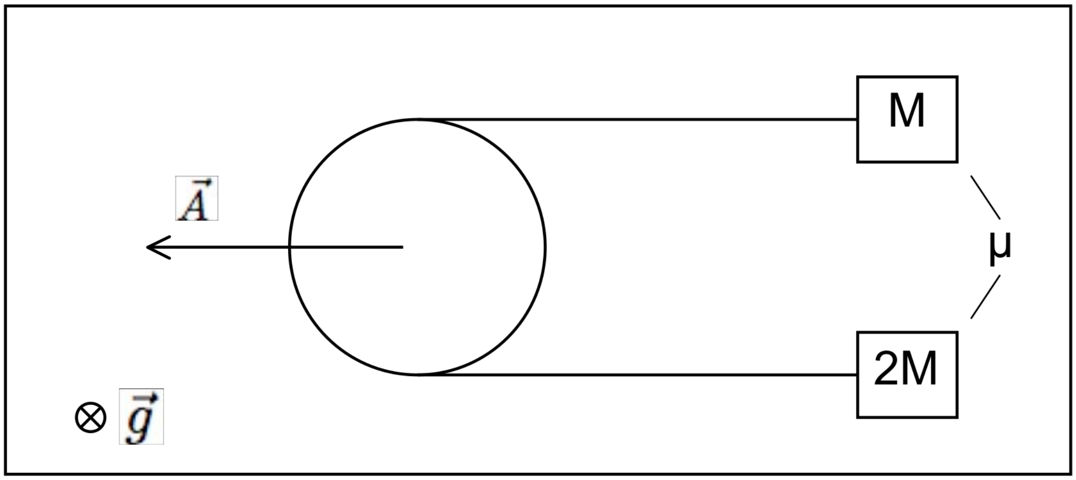


Less Greater The same Zero

(e) As a swinging pendulum passes through its equilibrium point, in which direction does the total net force act?

Radially outward Radially inward There is zero net force

Problem 2: The Accelerated Atwood Machine [20 pts]



An idealized Atwood machine (massless pulley and string) connected to two blocks of masses M and 2M sits initially at rest on a flat horizontal table. The coefficient of static and kinetic friction (assumed equal) between the block and table surfaces is μ. The pulley is accelerated to the left with magnitude of acceleration A. Assume that gravity acts with constant acceleration g down through the plane of the table.

(a) [15 pts] Find the distances each of the two blocks travel from their initial resting points as a function of time.

(b) [5 pts] What is the maximum acceleration A for which the block of mass 2M will remain stationary? Is there any case for A > 0 in which this block moves to the right?

SOLUTION

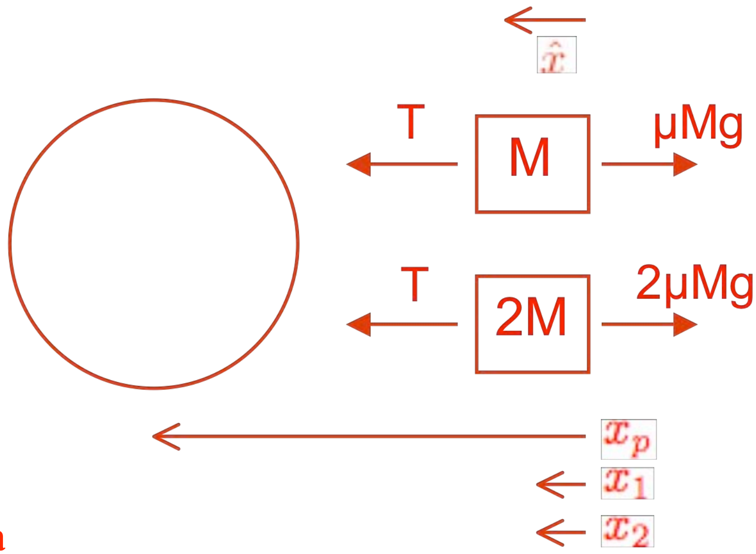


Fig 5a

(a) The geometry of the problem and force diagrams on for the blocks are shown above. Note that because the pulley is accelerating, we must keep track of its position in the inertial, one-dimensional coordinate system indicated above. Friction acts in the negative x direction and is simply equal to the normal force of the blocks on the surface of the table (their weight) time μ. The tension force from the string is the same for both blocks since the string is massless. Hence, the equations of motion for the two blocks are:

$$M\ddot{x}_1 = T - \mu M g$$

$$2M\ddot{x}_2 = T - 2\mu M g$$

The constraint equation is the length of the string, L, which must be constant. The string length can be computed as (R is the radius of the pulley wheel):

$$x_p - x_1 + x_p - x_2 + \pi R = L$$

$$\Rightarrow \ddot{x}_1 + \ddot{x}_2 = 2\ddot{x}_p = 2A$$

Taking the equations of motion, dividing by the lead mass terms and adding the equations together gives us:

$$\ddot{x}_1 + \ddot{x}_2 = 2A = \frac{3T}{2M} - 2\mu g$$

$$\Rightarrow T = \frac{4}{3}MA + \frac{4}{3}\mu M g$$

And hence:

$$\ddot{x}_1 = \frac{4}{3}A + \frac{4}{3}\mu g - \mu g = \frac{1}{3}(4A + \mu g)$$

$$\ddot{x}_2 = \frac{2}{3}A + \frac{2}{3}\mu g - \mu g = \frac{1}{3}(2A - \mu g)$$

These are constant acceleration terms, so the positions of the two masses is readily determined (assuming $x_1(0) = x_2(0) = 0$ and that the blocks are initially at rest):

$$\boxed{\begin{aligned} x_1(t) &= \frac{1}{6}(4A + \mu g)t^2 \\ x_2(t) &= \frac{1}{6}(2A - \mu g)t^2 \end{aligned}}$$

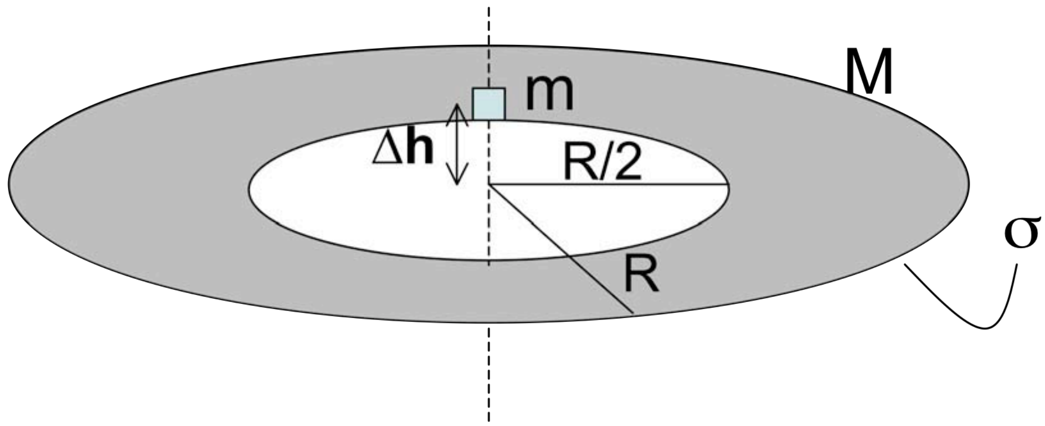
This is only part of the solution, since if A is insufficient to accelerate the larger block (see next part), then $x_2(t) = 0$

(b) Based on the equations above, the larger mass will not move if:

$$\boxed{A \leq \frac{1}{2}\mu g}$$

The second mass **cannot move backward** – the friction forces exactly matches the tension from the string for A < μg/2.

Problem 3: Oscillation through a Massive Disk [25 pts]



A small block of mass m lies above a thin disk of total mass M, constant surface density σ, and radius R. A hole of radius R/2 is cut into the center of the disk, allowing the mass to pass through. The mass starts at a position along the central axis of the disk but is displaced a height Δh from the plane of the disk, where Δh << R. There are no other external forces at work on this system; i.e., the disk and block are far out in deep space. Assume m << M.

(a) [15 pts] Show that to first order in Δh the block undergoes simple harmonic motion with respect to the center plane of the disk, and compute the period of oscillation of the block.

(b) [10 pts] What is the period of oscillation of the disk, and what is the ratio of the amplitude of the disk's oscillation to that of the block?

SOLUTION

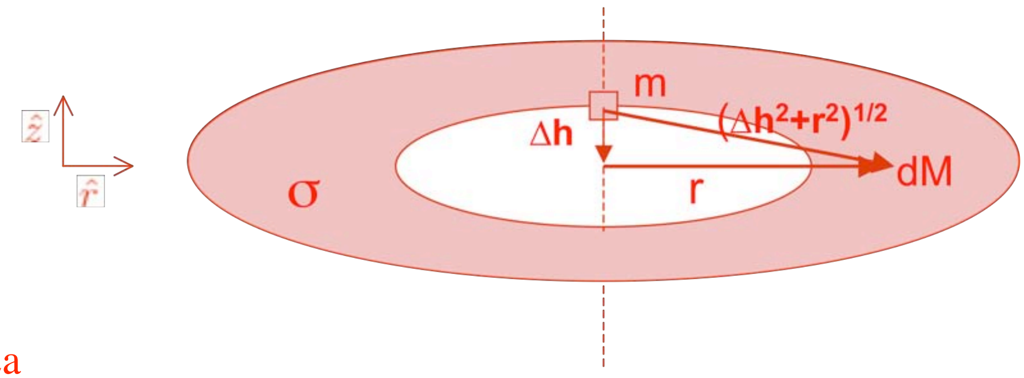


Fig 2a

(a) We break this problem up by considering the gravitational force acting on the block from one part of the disk dM, and then integrating over the entire disk. From the symmetry of the problem, we need only worry about the vertical component of the gravitational force. The geometry of the problem is illustrated above. The vertical component of force acting from dM is:

$$dF_z = -G\frac{mdM}{(\Delta h^2 + r^2)^{3/2}}\Delta h$$

$$F_z = -Gm\Delta h\int_V \frac{dM}{(\Delta h^2 + r^2)^{3/2}}$$

$$= -Gm\Delta h\int_{R/2}^{R}\int_0^{2\pi}\frac{\sigma r dr d\theta}{(\Delta h^2 + r^2)^{3/2}}$$

$$= -\pi Gm\sigma\Delta h\int_{R/2}^{R}\frac{2rdr}{(\Delta h^2 + r^2)^{3/2}}$$

Now make the substitution $u = \Delta h^2 + r^2$ and hence $du = 2rdr$, then integral can then be solved as:

$$\int_{u(R/2)}^{u(R)} \frac{du}{u^{3/2}} = -2u^{-1/2}\Big|_{u(R/2)}^{u(R)}$$
$$= 2[(\Delta h^2 + (R/2)^2)^{-1/2} - (\Delta h^2 + (R)^2)^{-1/2}]$$

The problem asks to solve this problem to first order in Δh so we can Taylor expand the solution to the integral as:

$$\approx 2[\frac{2}{R} - \frac{1}{2}\left(\frac{2}{R}\right)^3 \Delta h^2 - \frac{1}{R} - \frac{1}{2}\left(\frac{1}{R}\right)^3 \Delta h^2] \approx \frac{2}{R}$$

Hence, the z-component of gravitation force acting on the mass is, to first order in Δh,

$$F_z = m\ddot{\Delta h} = -\frac{2\pi G m \sigma}{R}\Delta h$$
$$\Rightarrow \boxed{\ddot{\Delta h} = -\frac{2\pi G \sigma}{R}\Delta h = -\omega^2 \Delta h}$$

Which is the equation of simple harmonic motion with period

$$\boxed{P = \frac{2\pi}{\omega} = \sqrt{\frac{2\pi R}{G\sigma}} = \pi\sqrt{\frac{3R^3}{2GM}}}$$

(b) The force exerted on the block by the disk must be equal and opposite to the force exerted on the disk by the block; hence:

$$M\ddot{z}_{disk} = \frac{2\pi G m \sigma}{R}\Delta h$$

The disk should have the same oscillation period as the block as it is always pulled back toward the block when the block is on either side. There is no way for the oscillations to be "out of phase" since there is no third body to induce a new frequency. The equation above does not quite look like it is going to give the same frequency. However, the variables z_{disk} and Δh are not the same, describing the offset of the disk with respect to some fixed midplane and the offset of the block with respect to the disk (we didn't take into account the fact that Δh is a non-inertial coordinate above, but in fact the correction is very small if $m << M$) . The maxima of Δh and z_{disk} are the amplitudes of oscillation for the block and disk, and are exactly 180° out of phase with each other. Hence for the above equation to give us the same harmonic motion, we must have:

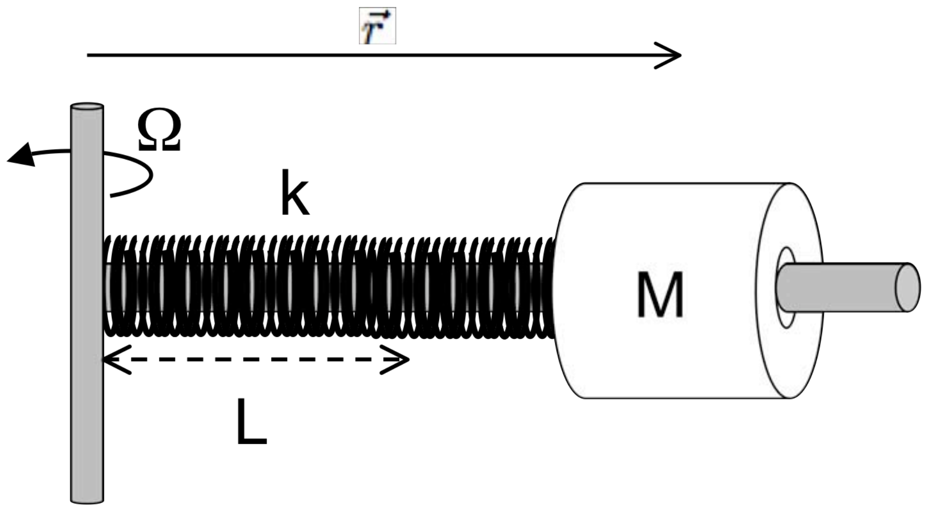
$$\boxed{z_{disk} = -\frac{m}{M}\Delta h}$$

i.e., the amplitude of the disk is scaled by a factor m/M to the amplitude of the block.

Another way to approach this part is to note that the system is isolated so the center of mass must remain fixed. This means that the oscillations must be in phase (so the periods are the same) and that:

$$R_{COM} = \frac{m\Delta h + Mz_{disk}}{m+M} = 0 \Rightarrow z_{disk} = -\frac{m}{M}\Delta h$$

Problem 4: Rotating Spring [20 pts]



A cylindrical mass M is placed on a post connected to a rotating shaft. The post forces the mass to rotate with the shaft at constant angular velocity Ω. The mass is connected to the shaft by a spring with spring constant k and rest length L. Ignore friction and gravity.

(a) [5 pts] What is the equilibrium distance of the mass from the central shaft?

(b) [10 pts] At time t=0, the mass is struck, giving it an inward radial velocity V_R. Derive the period of oscillation of the cylindrical mass and an expression for its distance from the central shaft as a function of time t.

(c) [5 pts] What happens when $\Omega^2 > k/M$?

SOLUTION

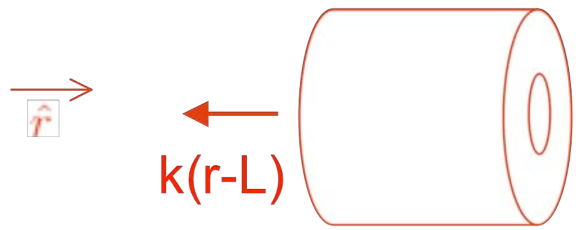


Fig 3a

(a) Because of the constraint of the post, we are only interested in radial motion. The only identifiable force acting on the system is the restoring spring force. Hence, writing the equations down in radial coordinates, and assuming r is fixed at r_{eq},

$$M(\ddot{r} - r\dot{\theta}^2) = -M r_{eq}\Omega^2 = -k(r_{eq} - L)$$

$$\boxed{\Rightarrow r_{eq} = \frac{k}{k - M\Omega^2}L}$$

(b) As r is no longer fixed, the equations of motion become:

$$M\ddot{r} = Mr\Omega^2 - k(r - L) \Rightarrow \ddot{r} = -(\frac{k}{M} - \Omega^2)r + \frac{kL}{M}$$

Make the substitution

$$r' = r - \frac{kL}{M}\left(\frac{M}{k - M\Omega^2}\right) = r - r_{eq}$$

$$\ddot{r}' = \ddot{r}$$

and hence

$$\ddot{r}' = -(\frac{k}{M} - \Omega^2)r' = -\omega^2 r'$$

This is the general expression for simple harmonic motion where

$$\omega^2 = \frac{k}{M} - \Omega^2$$

and hence

$$\boxed{P = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{M}{k - M\Omega^2}} = 2\pi\sqrt{\frac{M r_{eq}}{kL}}}$$

The solution with time is:

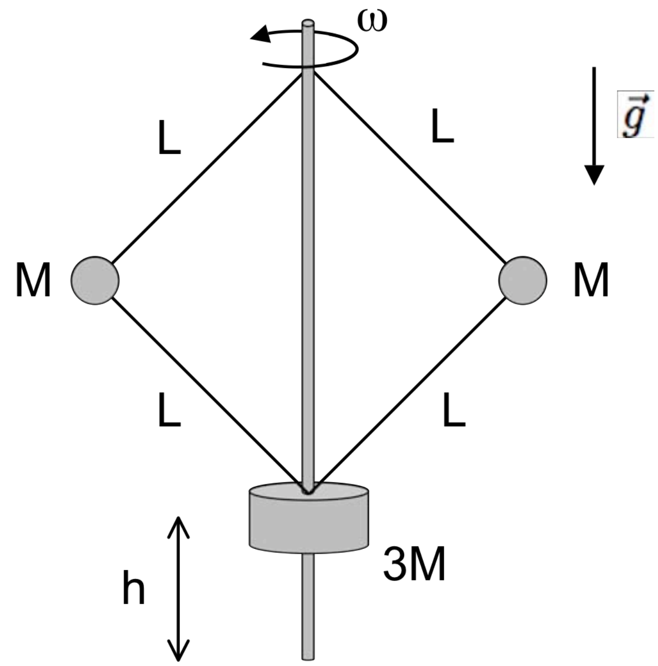
$$r(t) = r'(t) + r_{eq} = A\sin\omega t + B\cos\omega t + r_{eq}$$

The initial conditions are $r(0) = r_{eq}$ and $\dot{r}(0) = -V_r$, hence B=0 and $A\omega = -V_r$, so

$$\boxed{r(t) = r_{eq} - \frac{V_r}{\omega}\sin\omega t}$$

(c) If Ω^2 approaches k/M, the equilibrium length tends toward ∞ and the period tends toward 0, indicating a situation in which the spring becomes "fully stretched". This limit cannot be realized in nature; before this happens tension forces between the molecules of the spring will keep it intact, unless the system rotates so fast that the spring snaps or detaches from the central shaft.

Problem 5: The Flyball Governor [25 pts]



A flyball governor is a device commonly used in steam engines to control the flow of steam. In the simplified version shown above, a rotating shaft is connected to two hinges of mass M through rigid, massless rods of length L. The rods are also attached at the bottom of the device to a larger block of mass 3M which can slide freely up and down the shaft (in a practical flyball governor, this mass would be attached to a valve closer to reduce fuel flow). The shaft rotates at constant angular rate ω. Assume constant gravitational acceleration g acting downward and ignore friction and viscosity effects.

(a) [20 pts] Derive, as a function of ω, the height of the large block above its non-rotating position.

(b) [5 pts] Derive an expression relating the vertical speed of the block to the time rate change in angular rotation rate of the shaft, and confirm that the block moves upward as the shaft spins faster.

SOLUTION

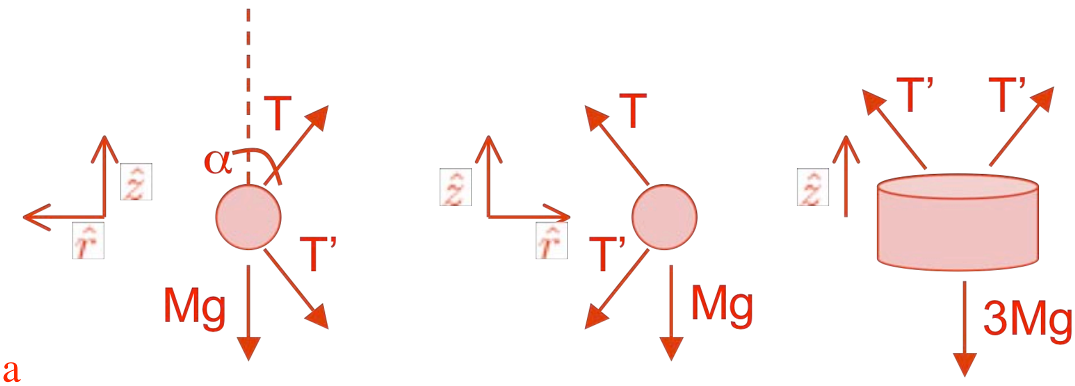


Fig 5a

(a) Force diagrams for each of the hinges are shown above. Assume that the angle between the rods and the vertical is α, so that the height of the block is:

$$h = 2L(1 - \cos\alpha)$$

and the radial distance of the hinges from the central shaft is

$$R = L\sin\alpha$$

We adopt a cylindrical coordinate system, ignoring the $\hat{\theta}$ direction. For each hinge the equations of motion are identical:

$$\hat{r} : M(\ddot{r} - r\dot{\theta}^2) = -ML\sin\alpha\omega^2 = -T\sin\alpha - T'\sin\alpha$$
$$\hat{z} : M\ddot{z} = T\cos\alpha - T'\cos\alpha - Mg = 0$$

The unknown variable T' can be found by dividing both equations though by the appropriate trignometric function of α and adding them together, yielding:

$$2T' = ML\omega^2 - \frac{Mg}{\cos\alpha}$$

We are only concerned about the z component of motion for the block, and this equation of motion is:

$$3M\ddot{z} = 2T'\cos\alpha - 3Mg = (ML\omega^2\cos\alpha - Mg) - 3Mg = 0$$
$$\Rightarrow \cos\alpha = \frac{4g}{L\omega^2}$$

hence

$$\boxed{h = 2L(1 - \frac{4g}{L\omega^2})}$$

**Note that this expression only applies for a minimum value of ω; i.e., h cannot be negative. This minimum values corresponds to the rotation speed required for the object to lift up.

(b) An expression for the vertical speed of the block and be derived simply by taking the derivative of our solution to part (a):

$$\boxed{\frac{dh}{dt} = \frac{16g}{\omega^3}\frac{d\omega}{dt} = \frac{16g\alpha}{\omega^3}}$$

Since positive dh/dt corresponds to motion upwards, then an increase in spin rate (dω/dt > 0) does correspond to lift in the block.

** Note that this assumes slow increases in h and/or ω - if the block accelerates appreciably, the rods may also push out the hinges, and an oscillation may insue. Note also that, again, ω cannot equal 0 so the infinite solution is avoided.

USEFUL EQUATIONS

Trajectory for constant acceleration $\vec{a}$

$$\vec{r}(t) = \frac{1}{2}\vec{a}t^2 + \vec{v}(0)t + \vec{r}(0)$$

Velocity in polar coordinates

$$\frac{d\vec{r}}{dt} = \dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

Acceleration in polar coordinates

$$\frac{d^2\vec{r}}{dt^2} = \ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$

Taylor expansion of function f(x):

$$f(x) = f(a) + \frac{1}{1!}\frac{df}{dx}|_a(x-a) + \frac{1}{2!}\frac{d^2f}{dx^2}|_a(x-a)^2 + ...$$