Lecture 4: Toroidal Equilibrium and Radial Pressure Balance
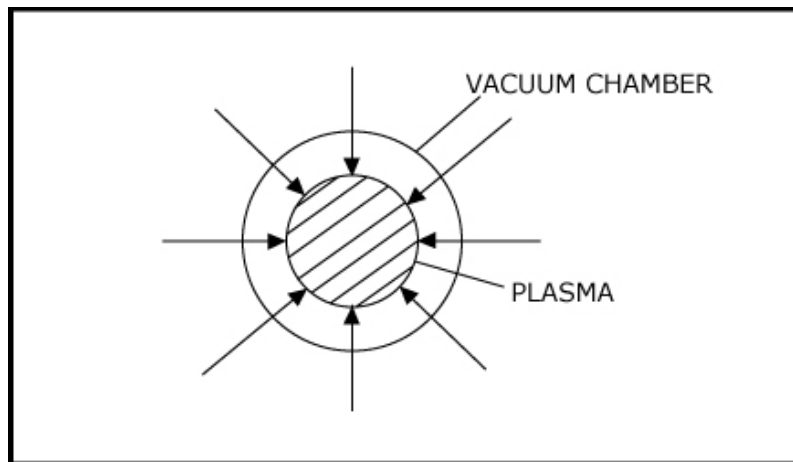
Basic Problem of Toroidal Equilibrium

1. Radial pressure balance

2. Toroidal force balance

Radial Pressure Balance

1. The largest forces are usually associated with radial pressure balance

2. The magnetic field must confine the plasma radially so it is isolated from the vacuum chamber.

3. There are several ways to do this using toroidal and/or poloidal fields. This is not very difficult to accomplish.
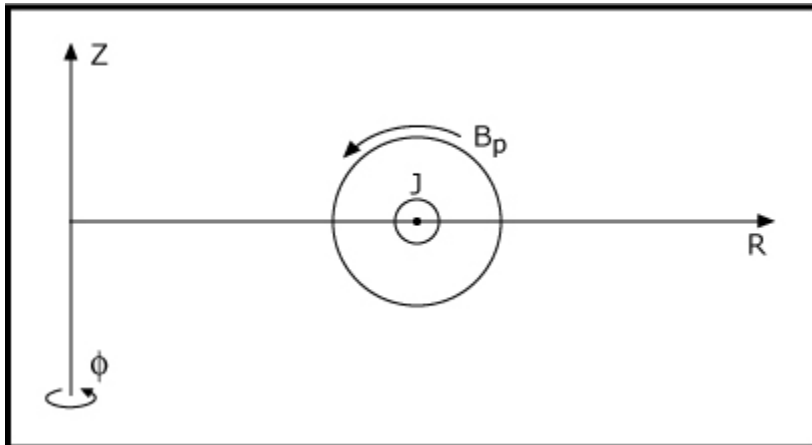
Toroidal Force Balance

1. Smaller forces are also present associated with unavoidable outward toroidal expansion forces
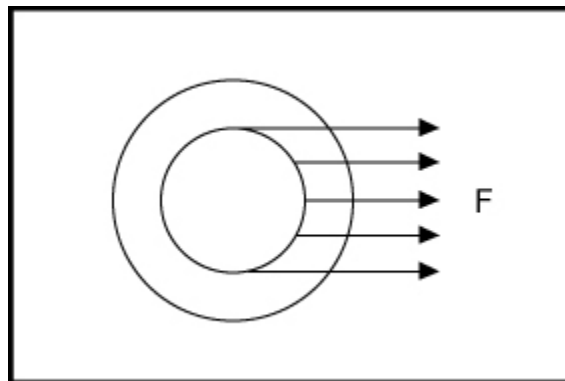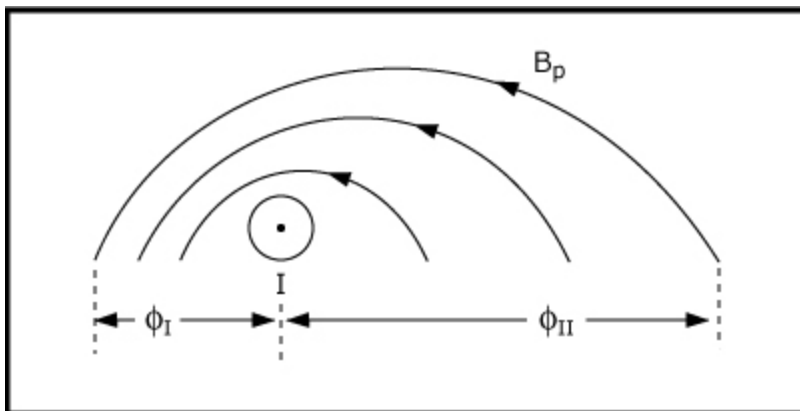
2. We now consider two opposing limits to demonstrate the basic problem of toroidal equilibrium. Even though the forces are smaller, toroidal force balance is the crucial issue.
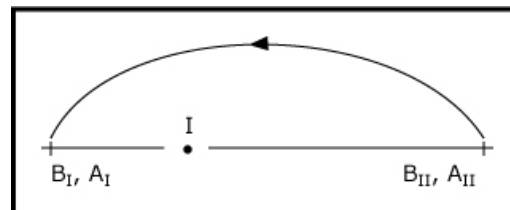
Purely Poloidal Field

1.

2. Outward Force #1

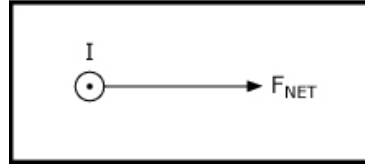   a. flux conservation: $\phi_I = \phi_{II}$ or $B_I A_I \sim B_{II} A_{II}$

    b. fields: $B_I > B_{II}$
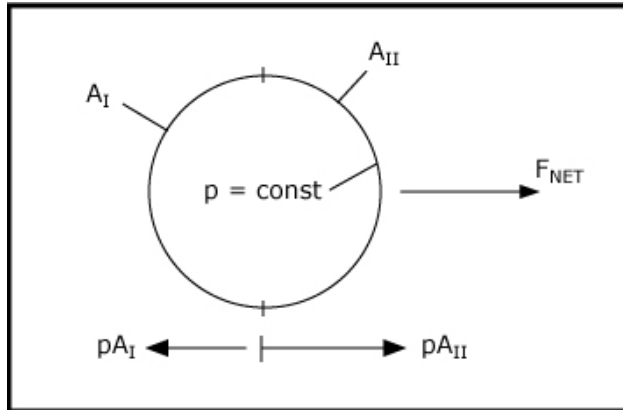
       areas: $A_I < A_{II}$

    c. force: $B_I^2 A_I > B_{II}^2 A_{II}$

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d. this leads to net outward *hoop force*

3. Outward force #2

a. The pressure force is outward. It results from a larger area on the outside compared to the inside at fixed pressure. This is similar to a *rubber tire tube force.*

4. Restoring force #1

a. plasma shifts outward

b. flux is compressed against the perfectly conducting shell

c. B_p increases on the outside of the torus

d. The increased magnetic pressure produces a force to balance the outward toroidal force. This is a *conducting shell equilibrium.*

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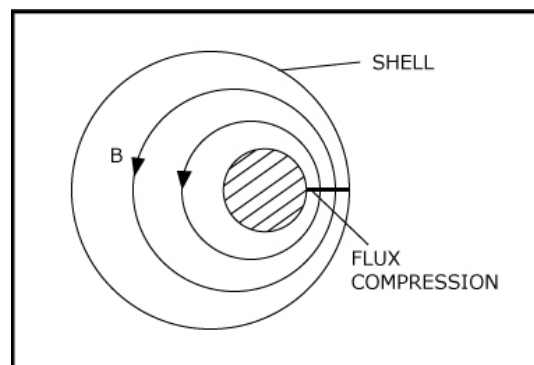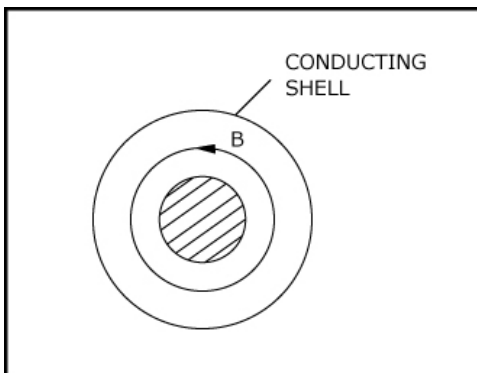
5. Restoring force #2

    a. A vertical field produces a $J \times B$ forces

    b. With the proper magnitude and sign, B_v can balance the outward force

   c. This is the *vertical field force* $F_v = B_v I_p L$
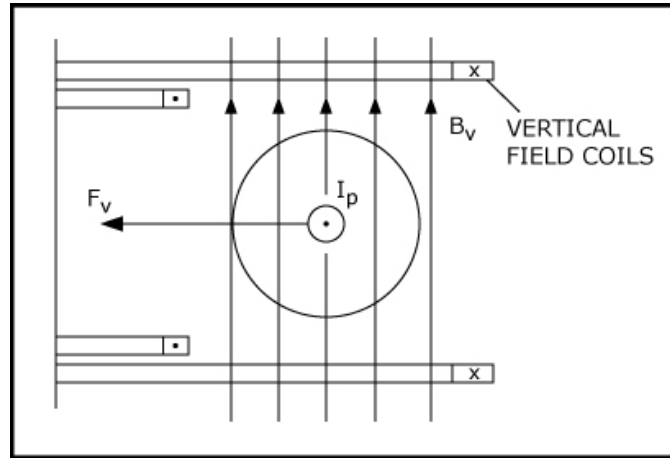
6. Conclusion: It is easy to provide toroidal equilibrium using purely poloidal magnetic fields

7. But, we shall see that such systems are very unstable to macroscopic MHD modes (in the purely poloidal case)
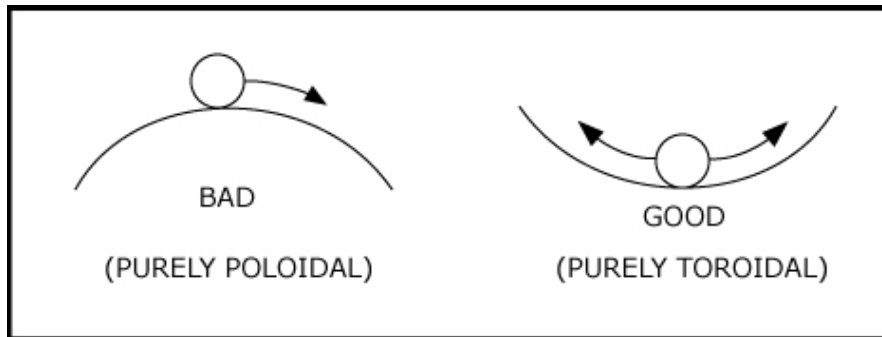
8. However, we shall also see that systems with large toroidal fields have much better stability properties.

Purely Toroidal Field

1. There is no hoop force since $I_{TOR} = 0$

2. The rubber tire tube force is present for the same reasons as before

3. There is an additional force due to J_p

4. Outward force #3
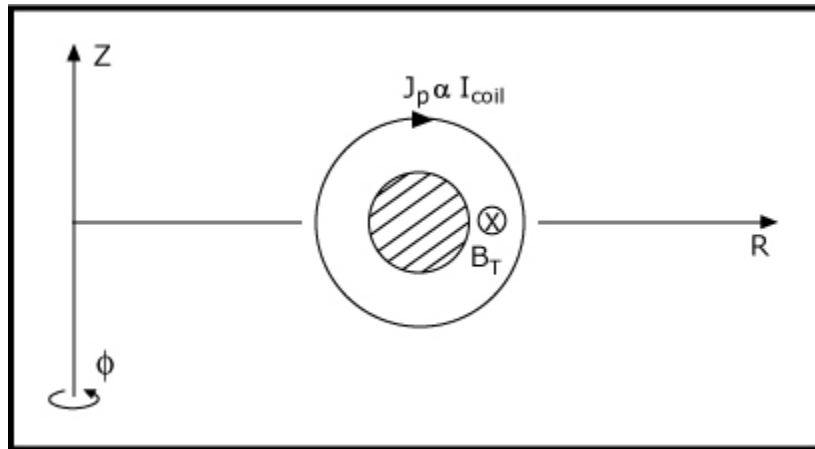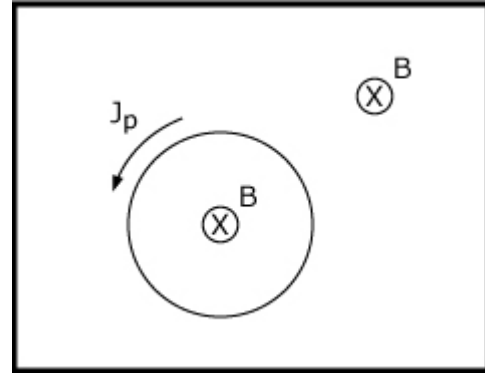
5. Note: $B_T \propto 1/R$: $\int B \cdot dl = \mu_0 I_{coil}$: $2\pi B_T R = \mu_0 I_{coil}$
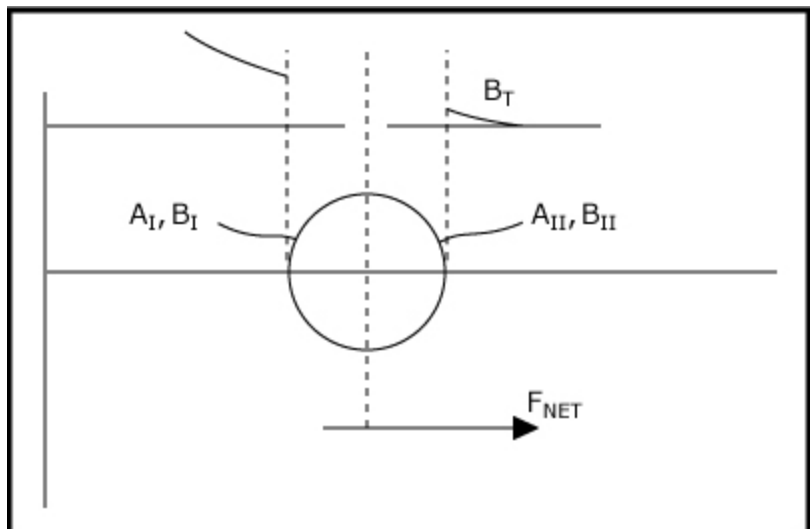
$$B_T = \frac{\mu_0 I_{coil}}{2\pi R}$$

6. Now
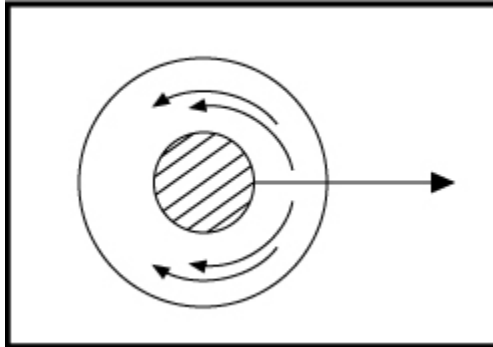
$$B_I > B_{II}$$

$$A_I < A_{II}$$

$$B_I^2 A_I > B_{II}^2 A_{II}$$

Since B^2 dominates, there is a net outward force
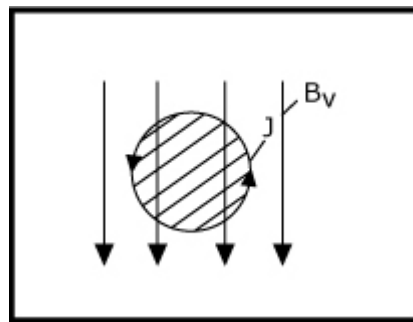
7. This is called the $1/R$ force

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8. Can a conducting shell balance outward force in purely toroidal case? No!

Magnetic flux is not trapped. Lines are free to slide around plasma

9. Can a vertical field balance outward force in purely toroidal case? No!

There is no net inward force because of the basic field directions
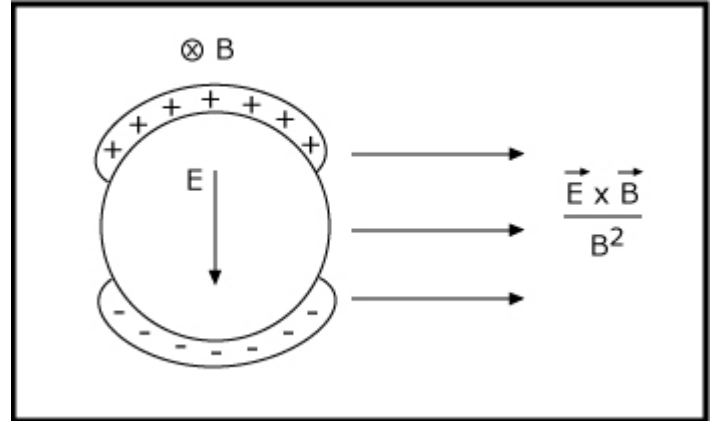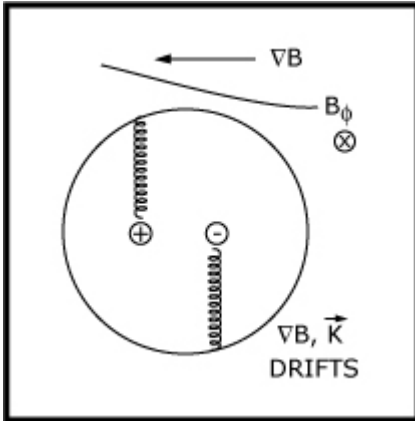
10. Conclusion

A purely toroidal field cannot hold a plasma in toroidal equilibrium. The toroidal force cannot be balanced.

11. The time scale for loss of equilibrium is in the $\mu \sec$ range.

For a = 1 m, B = 5 T, n = 10^{21} m^{-3}, $\beta = 1$, $R = 1\,\mathrm{km} \to t_{equil} = 18\mu\sec$.

12. Single particle picture

13. Effect of transform $\rightarrow$ poloidal field
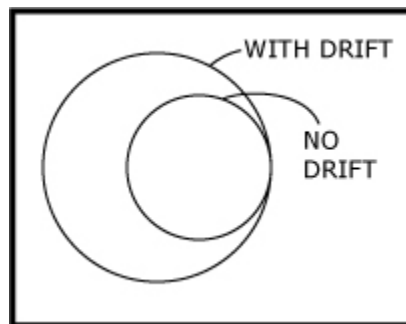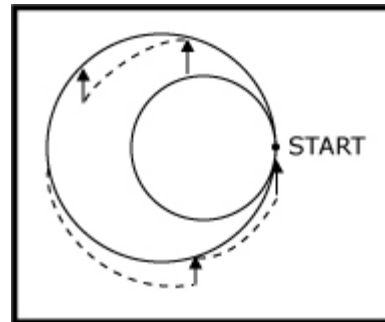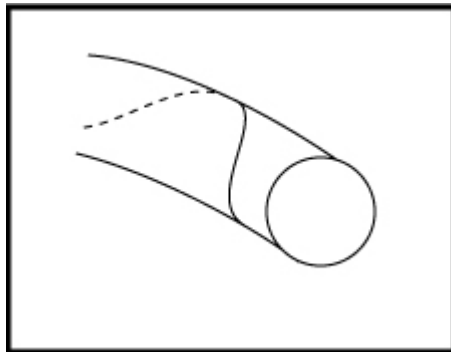
Basic Problem of Toroidal Equilibrium

1. Poloidal fields: Good equilibrium $\rightarrow$ poor stability

2. Toroidal fields: Poor equilibrium $\rightarrow$ good stability

Our goal is to optimize the advantages and minimize the disadvantages. This is the challenge of creating desirable fusion geometries.

General Approach to MHD Equilibria

1. The MHD equilibrium problem separates into two parts for most configurations of interest:

    a. radial pressure balance: zero order in a/R

   b. toroidal force balance: first order in a/R

2. We examine radial pressure balance in a 1-D geometry: straight cylinder (no problems with toroidal force balance)

3. The 1-D radial pressure balance relation is valid for tokamaks, stellarators, RFP's, pinches, and EBT's

4. We introduce toroidal effects as an aspect ratio expansion to see what must be done to achieve toroidal force balance. This requires 2-D calculations.

Radial Pressure Balance

1-D θ pinch

1. θ pinch $\rightarrow$ analog of the purely toroidal case.

2. Configuration
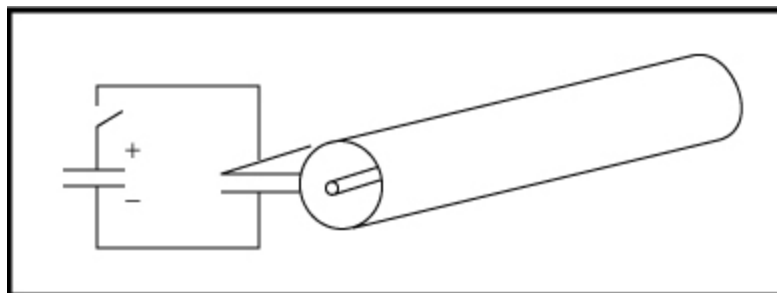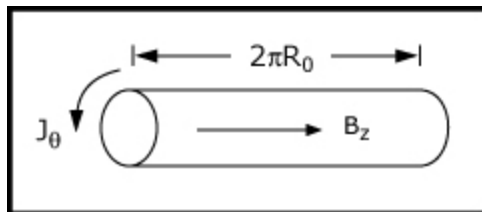
3. Nonzero components B= B_z $(r)e_z$, $p= p(r)$, J=J(r) e_θ

4. Solution of the MHD equilibrium equations

    a. $\nabla \cdot B = 0$ $\partial B_z/\partial z = 0$ automatically

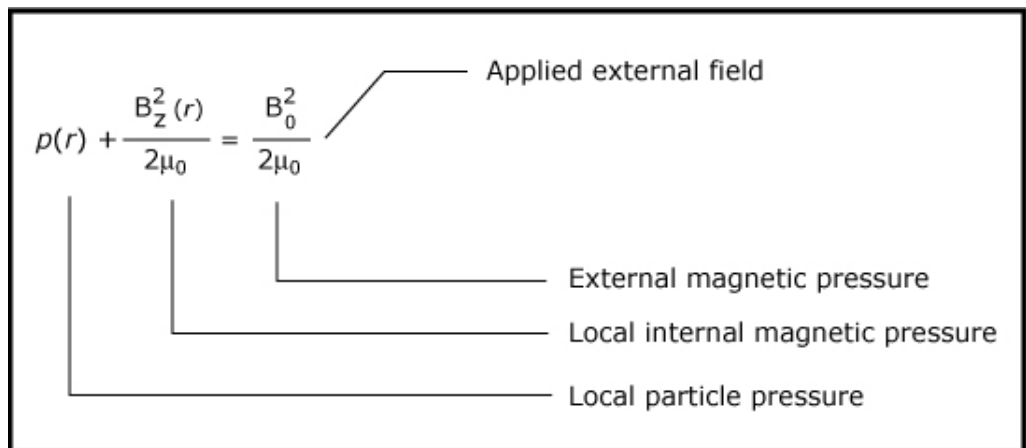b. $\mu_0 J = \nabla \times B \qquad \mu_0 J_\theta = -\partial B_z/\partial r$

c. $J \times B = \nabla p = (\partial p/\partial r) e_r$

$$J \times B = J_\theta B_z e_r = -(B_z/\mu_0) \partial B_z/\partial r\, e_r = -\partial\left(B_z^2/2\mu_0\right)\!\big/\partial r\; e_r$$

d. θ pinch pressure balance relation

$$\frac{\partial}{\partial r}\left(p + \frac{B_z^2}{2\mu_0}\right) = 0$$

5. Integrate

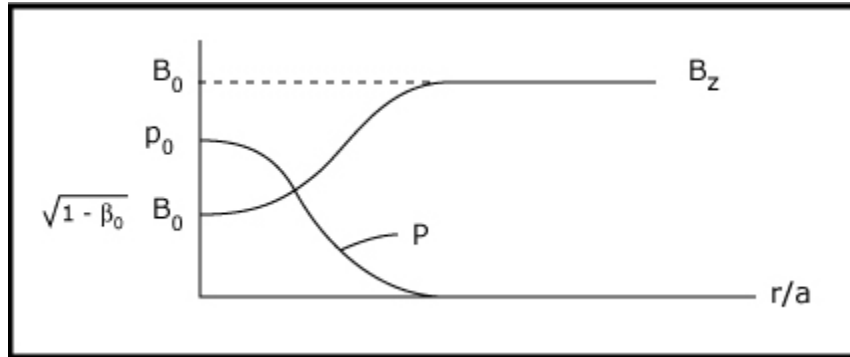6. Example: We are free to give one function arbitrarily, say $p(r)$

$$p(r) = p_0 e^{-r^2/a^2}$$

Then

$$\frac{B_z^2}{B_0^2} = 1 - \frac{2\mu_0 p_0}{B_0^2} e^{-r^2/a^2}$$

Define $\beta_0 \equiv 2\mu_0 p_0 / B_0^2$ "β on axis"

$$\frac{B_z^2(r)}{B_0^2} = 1 - \beta_0 e^{-r^2/a^2}$$

Note $0 < \beta_0 < 1$ for real solutions

7. In a θ pinch any value of β is possible (good for fusion) including very high β, $\beta = 1$

8. Concepts in which radial pressure balance corresponds to a θ pinch

   a. θ pinch

   b. stellarator

   c. high β stellarator

   d. high β tokamak

   e. EBT

   f. central cell of a tandem mirror

1-D Z Pinch

1. Z pinch – analog of the purely poloidal case

2. Configuration
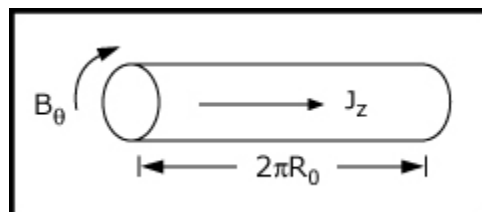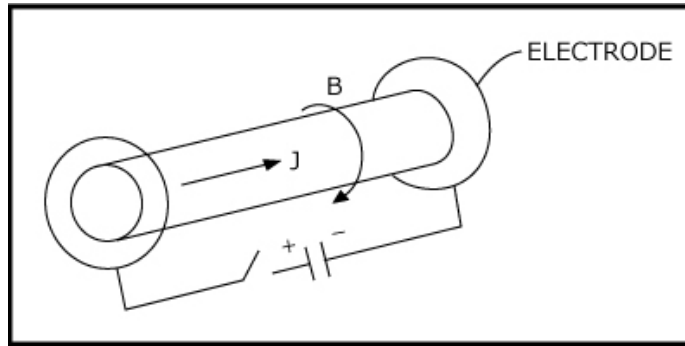
3. Nonzero components: $B = B_\theta(r)e_\theta$, $p = p(r)$, $J = J_z(r)e_z$

4. Solution of MHD equilibrium equations

    a. $\nabla \cdot B = 0$ $(1/r)(\partial B_\theta/\partial\theta) = 0$ automatically satisfied

    b. $\mu_0 J = \nabla \times B$ $\mu_0 J_z = (1/r)(\partial(rB_\theta))/\partial r$

    c. $J \times B = \nabla p$

    $$\nabla p = \frac{\partial p}{\partial r} e_r$$

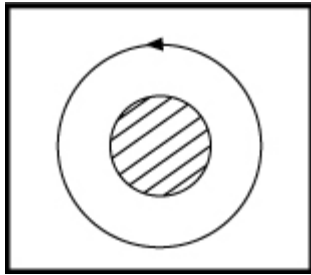 $$J \times B = -J_z B_\theta e_r = -\frac{B_\theta}{\mu_0 r}\frac{\partial}{\partial r} rB_\theta$$

    d. Z pinch pressure balance relation

    $$\frac{\partial p}{\partial r} + \frac{B_\theta}{\mu_0 r}\frac{\partial}{\partial r} rB_\theta = 0$$
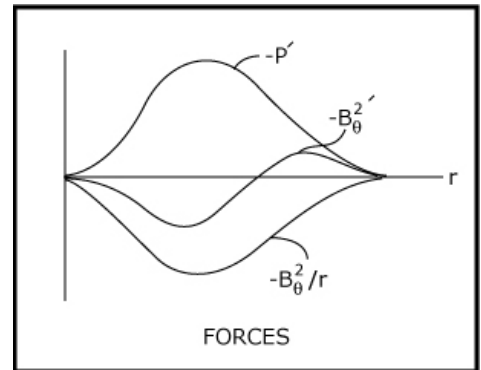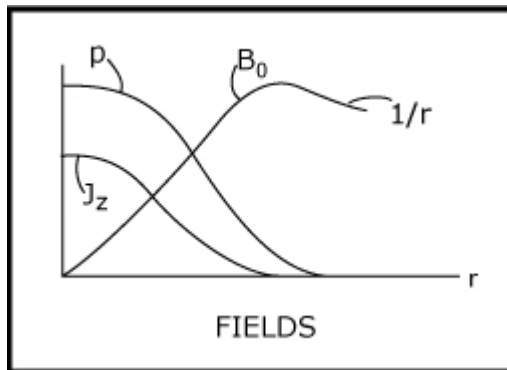
5. Alternate form

$$\frac{\partial}{\partial r}\left(p + \frac{B_\theta^2}{2\mu_0}\right) = \frac{B_\theta^2}{\mu_0 r} = 0$$

tension force

magnetic pressure force

particle pressure force

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Tension causes field lines to collapse. This produces an inward radial force

6. Typical profiles: for large $r, B_\theta \sim 1/r \rightarrow J_z = 0$

FIELDS

FORCES

7. The plasma is confined by magnetic tension. The magnetic pressure and plasma pressure each produce positive (outward forces) near the outside of the plasma.
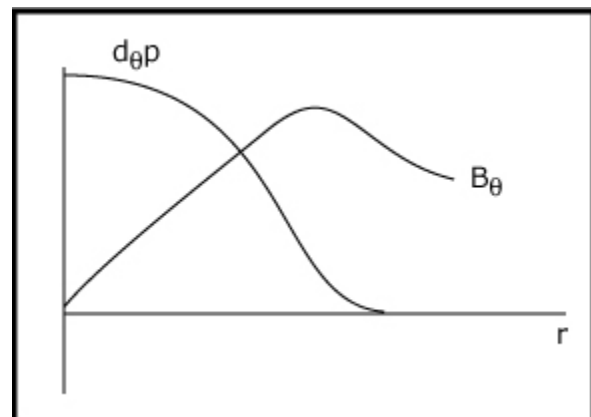
8. Example (Bennett Profiles)

$$p = \frac{p_0}{\left(1 + r^2/a^2\right)^2}, \quad B_\theta = \frac{\mu_0 I}{2\pi} \frac{r}{r^2 + a^2} \rightarrow \left(2ap_0\right)^{1/2} = \frac{I}{2\pi}$$

Then

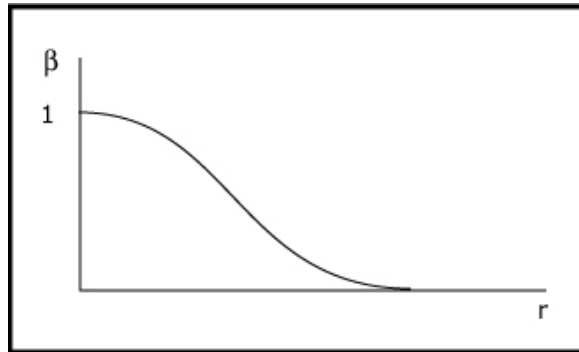$$B_\theta = \frac{\mu_0 I}{2\pi} \frac{r}{r^2 + a^2}$$

$$J = \frac{I}{\pi} \frac{a^2}{\left(r^2 + a^2\right)^2}$$

$$p = \frac{\mu_0 I^2}{8\pi^2} \frac{a^2}{\left(r^2 + a^2\right)^2}$$

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9. Define: $\beta(r) = 2\mu_0 p(r) / \left[2\mu_0 p(r) + B_\theta^2(r) \right]$

$$\beta(r) = \frac{1}{1 + r^2/a^2}$$

10. Note: $\beta(0) = 1$ is always true for a pure Z pinch. Although high β is good, it is more desirable to have some flexibility in order to avoid instabilities.

11. Bennett Pinch Relation

$$\int 2\pi pr \; dr = \frac{2\pi\mu_0 I^2}{8\pi^2} \int \frac{r \; dr \; a^2}{\left(r^2 + a^2\right)^2} = \frac{\mu_0 I^2}{4\pi} \frac{1}{2} \underbrace{\int_0^\infty \frac{dx}{\left(1 + x\right)^2}}_{1}$$

$$2\pi \int pr \; dr = \frac{\mu_0 I^2}{8\pi}$$

This, we shall see, is more general than only for the Bennett profiles

12. Configurations with Z pinch radial pressure balance

    a. ohmically heated tokamak

    b. reversed field pinch