22.615, MHD Theory of Fusion Systems
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Lecture 20

Resistive Wall Mode

1. We have seen that a perfectly conducting wall, placed in close proximity to the plasma can have a strong stabilizing effect on external kink modes.

2. In actual experiments, the metallic vacuum chamber surrounding the plasma is a good approximation to a perfectly conducting wall.

3. However, its conductivity is not infinite but is finite.

4. In fact we do not want the conductivity too high and/or, too thick because it would take too long externally applied feedback fields to penetrate the shell and interact with the plasma.

5. Also, higher resistivity, smaller currents are induced in the chamber during transients, alleviating power supply requirements.

6. The question raised here concerns the effect of finite resistivity of the wall on external kink stability.

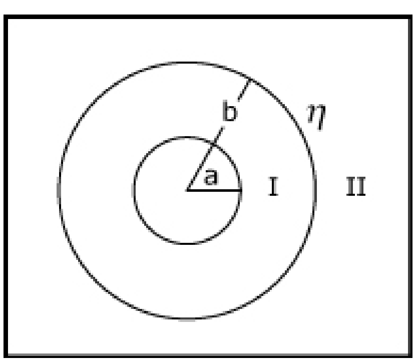
7. There are three possible situations and only one is really interesting.

8. In the first case the plasma is stable to external kinks with the wall at ∞ Here, since the plasma is already stable, a wall, either ideal or resistive does not affect stability. This case is uninteresting.

9. In the second case, the plasma is unstable with the wall at ∞ and with the wall at its actual position, assuming the wall is perfectly conducting. Since the plasma is unstable with a perfectly conducting wall as r=b, making the wall resistive does not help. This case is also uninteresting.

10. The interesting case is when the plasma is unstable with the wall at ∞, but stable with a perfectly conducting wall at r=b. Does the resistivity of the wall destroy wall stabilization?

11. To address this issue we investigate the problem in a straight cylindrical geometry. However, the results are valid for a general toroidal geometry as well.



Plan of attack

1. The analysis of the resistive wall mode is carried out in four steps.

2. First, reference values of δW are calculated for an ideal wall located as $\infty\left(\delta W_\infty\right)$ and at $r=b$ $\left(\delta W_b\right)$

3. The full eigenvalue problem is solved region by region assuming slow growing modes - on the scale of the wall diffusion time.

4. Third, the fields within the resistive wall are calculated using the then wall approximation. This gives rise to a set of jump conditions across the wall.

5. The resulting set of coupled equation and boundary conditions are solved yielding the dispersion relation.

The Reference Cases

1. Recall that δW for a general screw pinch surrounded by a perfectly conducting wall is given by

$$\frac{\delta W}{2\pi^2 R_0/\mu_0} = \int_0^a \left[f\xi'^2 + g\xi^2\right] dr + \left[\frac{F\hat{F}}{k_0^2} + \frac{r^2 \Lambda F^2}{|m|}\right]_a \xi_a^2$$

where $F = kB_z + \frac{mB_\theta}{r}$, $\hat{F} = kB_z - \frac{mB_\theta}{r}$

$$\Lambda \approx \frac{1+\left(a/b\right)^{2|m|}}{1-\left(a/b\right)^{2|m|}} \qquad k_0^2 = k^2 + \frac{m^2}{r^2}$$

2. The exact minimizing ξ satisfies

$$\left(F\xi'\right)' - g\xi = 0 \qquad \xi(a) = \xi_a \quad \xi(0) \text{ regular}$$

3. Recall that $F = \frac{rF^2}{k_0^2}$ and for an external mode with a resonant surface outside the plasma this implies that $F \neq 0$ in the plasma.

4. Thus the variational equation for ξ is non-singular. Its solution is important, but boring.

5. Assume the solution for ξ is known, either analytically or computationally.

6. If we multiply the equation for ξ by $\int_0^a (\ )\xi\, dr$ we find that

$$\int_0^a \left(F\xi'^2 + g\xi^2 \right) dr = F\xi\xi' \Big|_a$$

7. This allows us to write

$$\frac{\delta W}{2\pi^2 R_0/\mu_0} = \left[\frac{F\hat{F}}{k_0^2} + \frac{r^2 \Lambda F^2}{|m|} + \frac{F^2}{k_0^2}\left(\frac{r\xi'}{\xi}\right) \right]_a \xi_a^2$$

8. Note that $\left(r\xi'/\xi \right)_a$ is a known quantity from the solution for ξ.

9. The first reference case corresponds to the wall at $\infty : \Lambda = \Lambda_\infty = 1$
For this case

$$\frac{\delta W_\infty}{2\pi^2 R_0/\mu_0} = \left[\frac{F\hat{F}}{k_0^2} + \frac{r^2 F^2 \Lambda_\infty}{|m|} + \frac{F^2}{k_0^2}\left(\frac{r\xi'}{\xi}\right) \right]_a \xi_a^2$$

10. The second reference case corresponds to the wall at b

$$\Lambda = \Lambda_b = \left[1 + (a/b)^{2|m|} \right] \Big/ \left[1 - (a/b)^{2|m|} \right]$$

11. Keep in mind that $\Lambda_b > \Lambda_\infty$ (well stabilization)

12. For both reference cases $\left(r\xi'/\xi \right)_a$ is the same. It is unaffected by the wall.

13. These relations allow us to write

$$\frac{\delta W_b}{2\pi^2 R_0/\mu_0} = \frac{\delta W_\infty}{2\pi^2 R_0/\mu_0} + \left[\frac{r^2 F^2}{|m|} \right]_a \left(\Lambda_b - \Lambda_\infty \right) \xi_a^2$$

14. The interesting case under consideration corresponds to

$\delta W_\infty < 0$ unstable with the wall at ∞

$\delta W_b > 0$ stable with perfect wall at r=b

The eigenvalue problem with a resistive wall

1. We solve the full eigenvalue problem with the resistive wall

2. However, we can make use of much of what we have already done by assuming slow growing modes - resistive wall diffusion term.

3. Example: $a = .3$ m, $R_0 = 1$ m, $T_c = T_c - 2$ keV, $b \approx a$

4. Then $\tau_{MHD} = R_0/v_{T_c} = 2.3 \times 10^{-6}$ sec.

5. Consider a stainless steel vacuum chamber of thickness $d = 1$ mm. Then, with
   $\eta = 11 \times 10^{-8}\ \Omega m$
   $\tau_D = \mu_0 bd/\eta = 3.4 \times 10^{-3}$ sec.

6. For a thick copper wall d=1 cm, $\eta = 1.7 \times 10^{-8}\ \Omega m$.
   $\tau_D = \mu_0 bd/\eta = .22$ sec.

7. Clearly $\tau_D \gg \tau_{MHD}$ for either case.

8. The implication is that in the plasma eigenfunction equation,
   $\omega^2 \ll k_{\|}^2 v_a^2,\ \omega^2 \ll k_{\|}^2 v_{\tau_\iota}^2$ and $k_{\|} \neq 0$ for external mode. Therefore we can ignore ω^2 in the plasma region.

9. The resulting equation for ξ thus corresponds to the ideal marginal stability equation which is our old friend.

   $$\left(f\xi'\right)' - g\xi = 0$$

10. The ω's will appear where we discuss the wall.

11. The region between the plasma and the wall satisfies

    $$\tilde{B}_I = \nabla\phi_I, \nabla^2\phi_I = 0 \qquad \left(r\phi_I'\right)' - \left(k^2 + m^2/r^2\right)\phi_I = 0$$

12. The solution, neglecting k^2 for simplicity (to have polynomials rather than Bessel functions) is given by

    $$\phi_I = c_1\left(\frac{r}{b}\right)^{|m|} + c_2\left(\frac{b}{r}\right)^{|m|}$$

13. We will find c_1 and c_2 shortly by matching jump conditions

14. A similar analysis holds for the outer vacuum region where
    $\tilde{B}_{II} = \nabla\phi_{II} \quad \nabla^2\phi_{II} = 0$

15. The solution here has only a decaying solution since the fields must be regular as ∞. Thus

    $$\phi_{II} = c_3\left(\frac{b}{r}\right)^{|m|}$$

The wall solution

1. Now lets look within the wall

2. Assume the wall is then $d \ll b$. The wall looks rectangular

3. Let $r = b + x,\ \theta = y/b$

4. The equation for $\underline{\tilde{B}}$ in the wall is obtained as follows

$$\frac{\partial \underline{\tilde{B}}}{\partial t} = -\nabla \times \underline{\tilde{E}} = -\nabla \times \eta \underline{\tilde{J}} = -\nabla \times \frac{\eta}{\mu_0} \nabla \times \underline{\tilde{B}} = \frac{\eta}{\mu_0} \nabla^2 \times \underline{\tilde{B}}$$

5. Focus on the r (i.e. x component), and assume $\underline{\tilde{B}} \propto e^{-\iota\omega t} \propto e^{\omega\iota t}$
   $\omega_\iota =$ growth rate.

$$\frac{\partial \underline{\tilde{B}}_x}{\partial_x^2} - \left(k^2 + \frac{m^2}{b^2}\right)\tilde{B}_x = \frac{\mu_0 \omega_t}{\eta}\tilde{B}_x$$

6. $\tilde{B}_y$ and $\tilde{B}_z$ are found from $\nabla \cdot \underline{\tilde{B}} = 0$ and the assumption J_x=0 (then wall approx - all current flows parallel to the surface): $\underline{e}_x \cdot \nabla \times \underline{\tilde{B}} = 0$

7. We do not need $\tilde{B}_y$ and $\tilde{B}_z$ so we will not calculate them.

8. Then wall ordering: Assume $\omega_\iota \sim \dfrac{\eta}{\mu_0 bd} \sim \dfrac{J}{\tau_D}$

9. Then $\dfrac{\mu_0 \omega_\iota b^2}{\eta m^2} \sim \dfrac{\mu_0 b^2 \eta}{\eta\, bd} \sim \dfrac{b}{d} \gg 1$

10. Also $\tilde{B}_x^{\parallel} \Big/ \mu_0 \omega_\iota \tilde{B}_x\left(\eta\right) \sim \dfrac{1}{d^2}\dfrac{\eta}{\mu_0 \omega_\iota} = \dfrac{1}{d^2}\dfrac{\eta \mu_0 bd}{\mu_0 \eta} \sim \dfrac{b}{d} \gg 1$

11. This implies that the $\left(k^2 + \dfrac{m^2}{b^2}\right)\tilde{B}_x$ can be neglected and that $\tilde{B}_x = B_{x0} + B_{x1}\left(x\right)$ where $B_{X0} =$ const, $B_{x1}/B_{x0} \sim d/b \ll 1$

12. The equation and solution for B_{x1} are given by

$$\frac{\partial^2 B_{x1}}{\partial x^2} = \frac{\mu_0 \omega_\iota}{\eta} B_{x0}$$

$$B_x = B_{x0} + \frac{\mu_0 \omega_\iota}{\eta} B_{x0} \frac{x^2}{2}$$

13. For a thin wall $d/b \to 0$, this solution translates into the following two jump conditions

$$B_x\Big|_{b^-}^{b^+} = B_{x0} + \frac{\mu_0\omega_\iota}{\eta} B_{x0}\frac{d^2}{2} - B_{x0} \approx 0$$

$$B_x'\Big|_{b^-}^{b^+} = \frac{\mu_0\omega_\iota B_{x0} d}{\eta} \approx \frac{\mu_0\omega_\iota B}{\eta} \times d$$

14. Or $[\![B_r]\!] = 0 \quad [\![B_r']\!] = \dfrac{\mu_0\omega_\iota d B_r}{\eta}$

The jump condition and dispersion relation

1. There are four unknowns in the problem c_1, c_2, c_3, ω
2. There are four jump conditions. 2 at the wall given above, and 2 on the plasma we must now determine
3. The first is the usual $[\![\underline{n}\cdot\underline{B}]\!]$ condition

$$\underline{n}\cdot\underline{\tilde{B}}\Big|_a = \underline{n}\cdot\nabla\times\left(\underline{\xi}\times\underline{B}\right)\Big|_a$$

$$\boxed{\tilde{B}_{1r}\Big|_a = \iota F a \xi a}$$

4. The second is the pressure balance jump condition (lots of work)

$$\mu_0 p_1 + \underline{B}\cdot\underline{B}_1 + \underline{\xi}\cdot\nabla\left(\mu_0 p + \frac{\tilde{B}^2}{2}\right) = \underline{B}\cdot\underline{B}_1 + \underline{\xi}\cdot\nabla\frac{\tilde{B}^2}{2}$$

5. For no surface currents and p, p' as r=0 vanishing this reduces to

$$\underline{B}\cdot\nabla\times\left(\underline{\xi}\times\underline{B}\right)\Big|_a = \underline{B}\cdot\underline{\tilde{B}}_1\Big|_a$$

6. Vacuum part $\underline{B}\cdot\underline{\tilde{B}}_1 = \underline{B}\cdot\nabla\phi_I = \iota F\phi_I$
7. Plasma part $\underline{B}\cdot\nabla\times\left(\underline{\xi}\times\underline{B}\right) = \nabla\cdot\left(\underline{\xi}\times\underline{B}\right)\times\underline{B} - \underline{\xi}\times\underline{B}\cdot\nabla\times\underline{B}$ (last term $=0$ at the edge)

$$= -\nabla\cdot\underline{\xi_\perp}B^2 = -\underline{\xi_\perp}\cdot\nabla B^2 - B^2\nabla\cdot\underline{\xi_\perp}$$

8. Now $B^2 = B_z^2 + B_\theta^2$. Near the edge $B_z =$ const and $B_0 \sim \dfrac{k}{r}$. Therefore

$$\nabla B^2 : -\frac{2B_\theta^2}{r}\Big|_a e_r \text{ and } \underline{\xi_\perp}\cdot\nabla B^2 = -\frac{2\xi B_\theta^2}{r}$$

9. The last term is $B^2\nabla\cdot\underline{\xi_\perp}$ where

$$\nabla\cdot\underline{\xi_\perp}=\frac{1}{r}(r\xi)'+\nabla\cdot\underline{\eta}=\frac{1}{r}(r\xi)'+\nabla\cdot\left(\eta\frac{B_z e_0}{B}-\frac{B_\theta e_z}{B}\right)$$

$$=\frac{1}{r}(r\xi)'+\frac{\eta}{B}\left(\frac{\iota m B_z}{r}-\iota k B_\theta\right)=\frac{1}{r}(r\xi)'+\frac{\iota G}{B}\frac{\iota}{rk_0^2 B}\left(G(r\xi)'-2k_0 B_\theta\xi\right)$$

$$G=\frac{mB_z}{r}-kB_\theta \qquad \eta=\frac{\iota}{rk_0^2B}\left[G(r\xi)'+2kB_\theta\xi\right]$$

10. Note $\frac{1}{k_0^2B^2}\left(k_0^2B^2-G^2\right)=\frac{F^2}{k_0^2B^2}$

11. Combine term

$$\nabla\cdot\underline{\xi_\perp}=\frac{F^2}{k_0^2B^2}\frac{(r\xi)'}{r}-\frac{2kGB_\theta}{rk_0^2B^2}\xi$$

12. Collect term

$$\underline{B}\cdot\nabla\times(\xi\times B)=\frac{2B_\theta}{r}\xi-\frac{F^2}{k_0^2}\frac{(r\xi)'}{r}+\frac{2kGB_\theta}{rk_0^2}\xi$$

$$=-\frac{F^2}{k_0^2}\xi'+\left(\frac{2B_\theta^2}{r}-\frac{F^2}{rk_0^2}+\frac{2kGB_\theta}{rk_0^2}\right)\xi$$

$$=-\frac{F^2}{k_0^2}\xi'-\frac{F\hat{F}}{rk_0^2}\xi$$

13. Pressure balance boundary condition

$$\iota F\phi_I\Big|_a=-\frac{F^2}{k_0^2}\xi'-\frac{F\hat{F}}{rk_0^2}\xi\Bigg|_a$$

or

$$\phi_I\Big|_a=\frac{\iota}{rk_0^2}\left(\hat{F}+\frac{r\xi' F}{\xi}\right)\xi\Bigg|_a$$

Summary of where we are

$$\phi_I = c_1\left(\frac{r}{b}\right)^{|m|} + c_2\left(\frac{b}{r}\right)^{|m|}$$

$$\varphi_{II} = c_3\left(\frac{b}{r}\right)^{|m|}$$

$$\text{As } r = b \qquad \left[\!\left[\tilde{B}_r\right]\!\right] = 0, \left[\!\left[\tilde{B}_r^{'}\right]\!\right] = \frac{\mu_0\omega_\iota d\tilde{B}_r}{\eta}$$

$$\text{As } r = a \qquad \tilde{B}_{1r} = \iota F\xi_a,\ \phi_I = \frac{\iota}{rk_0^2}\left(\hat{F} + \frac{r\,\xi^{'}F}{\xi}\right)\xi_a$$

Apply B.C (note: $B_r = \varphi^{'}$**)**

$$\text{As } r = b \qquad \left[\!\left[\tilde{B}_{1r}\right]\!\right] = 0 \qquad \frac{|m|}{b}\left(c_1 - c_2\right) = \frac{|m|}{b}\left(-c_3\right)$$

$$\text{As } r = b \qquad \left[\!\left[\tilde{B}_r^{'}\right]\!\right] = \frac{\mu_0\omega_\iota d\tilde{B}_r}{\eta} - \frac{|m|}{b^2}\left[\left(|m|-1\right)c_1 + \left(|m|+1\right)c_2\right] + \frac{|m|\left(|m|+1\right)}{b^2}c_3 =$$

$$+\frac{|m|}{b}\left(c_1 - c_2\right)\frac{\mu_0\omega_\iota d}{\eta}$$

$$\text{As } r = a \qquad \tilde{B}_{1r}^{'} = \iota F\xi \qquad \frac{|m|}{b^2}\left[\frac{c_1}{W^{|m|-1}} - W^{m+1}c_2\right] = \iota F\xi a \quad W = \frac{b}{a}$$

$$\text{As } r = a \qquad \phi = \frac{\iota}{rk_0^2}\left(\tilde{F} + \frac{r\xi^{'}F}{\xi}\right)\xi \qquad \frac{c_1}{W^m} + c_2W^m = \frac{\iota}{rk_0^2}\left(\hat{F} + \frac{r\xi^{'}F}{\xi}\right)$$

Solve for c_1, c_2, c_3 **from 3 equations**

$$c_1 - c_2 + c_3 = 0$$

$$c_1 - W^{2m}c_2 = \frac{\iota aFW^m\xi}{m}$$

$$c_1 + W^{2m}c_2 = \frac{\iota aW^m}{a^2k_a^2}\left(\hat{F} + \frac{r\xi^{'}F}{\xi}\right)\xi$$

Solution

$$c_1 = \frac{\iota a W^m}{2}\left[\frac{F}{m} + \frac{\hat{F}}{k_a^2 a^2} + \frac{a\xi_a'}{\xi_a}\frac{F}{k_a^2 a^2}\right]\xi$$

$$c_2 = \frac{\iota a}{2W^m}\left[-\frac{F}{m} + \frac{\hat{F}}{k_a^2 a^2} + \frac{a\xi_a'}{\xi_a}\frac{F'}{k_a^2 a^2}\right]\xi$$

$$c_3 = c_2 - c_1 = -\frac{\iota a}{2W^m}\left[\left(W^{2m}+1\right)\frac{F}{m} + \left(W^{2m}-1\right)\frac{\hat{F}}{k_a^2 a^2} + \left(W^{2m}-1\right)\frac{F^2}{k_a^2 a^2}\frac{a\xi_a'}{\xi_a}\right]$$

Dispersion Relation (last equation)

$$-\left(m-1\right)c_1 - \left(m+1\right)c_2 + \left(m+1\right)c_3 = \left(c_1 - c_2\right)\frac{\mu_0 \omega_\iota db}{\eta}$$

Define $\tau_D = \mu_0 \, db/n$, set $c_3 = c_2 - c_1$

Then $\omega_\iota \tau_D = \dfrac{2c_1}{c_3}$

Simplify

$$\omega_\iota \tau_D = -\frac{2W^m\left[\dfrac{F}{m} + \dfrac{\hat{F}}{k^2 a^2} + \dfrac{a\xi_a'}{\xi_a}\dfrac{F}{k^2 a^2}\right]}{\dfrac{1}{W}\left(W^{2m}-1\right)\left[\dfrac{\hat{F}}{k^2 a^2} + \dfrac{F}{k^2 a^2}\dfrac{a\xi_a'}{\xi_a} + \dfrac{W^{2m}+1}{W^{2m}-1}\dfrac{F}{m}\right]}$$

Recall

$$\frac{\delta W_\infty}{2\pi^2 R_0 / \mu_0} = Fa^2\left[\frac{\hat{F}}{k_a^2 a^2} + \frac{F}{m} + \frac{F}{k_a^2 a^2}\frac{a\xi_a'}{\xi_a}\right]$$

$$\frac{\delta W_b}{2\pi^2 R_0 / \mu_0} = Fa^2\left[\frac{\hat{F}}{k_a^2 a^2} + \frac{F}{m}\left(\frac{W^{2m}+1}{W^{2m}-1}\right) + \frac{F}{k_a^2 a^2}\frac{a\xi_a'}{\xi_a}\right]$$

Therefore

$$\omega_\iota \tau_D = -\frac{2W^{2m}}{W^{2m}-1}\frac{\delta W_\infty}{\delta W_b}$$

Resistive wall mode is unstable!! $\delta W_\infty < 0 \quad \delta W_h > 0$

Growth rate $\sim 1/\tau_D$