22.101 Applied Nuclear Physics (Fall 2006)

Lecture 16 (11/8/06)

Neutron Interactions: Q-Equation, Elastic Scattering

References:

R. D. Evans, Atomic Nucleus (McGraw-Hill New York, 1955), Chap. 12.

W. E. Meyerhof, *Elements of Nuclear Physics* (McGraw-Hill, New York, 1967), Sec. 3.3.

Since a neutron has no charge it can easily enter into a nucleus and cause a reaction. Neutrons interact primarily with the nucleus of an atom, except in the special case of magnetic scattering where the interaction involves the neutron spin and the magnetic moment of the atom. Because magnetic scattering is of no interest in this class, we can neglect the interaction between neutrons and electrons and think of atoms and nuclei interchangeably. Neutron reactions can take place at any energy, so one has to pay particular attention to the energy variation of the interaction cross section. In a nuclear reactor neutrons can have energies ranging from 10^{-3} ev (1 mev) to 10^{7} ev (10 Mev). This means our study of neutron interactions, in principle, will have to cover an energy range of 10 ten orders of magnitude. In practice we will limit ourselves to two energy ranges, the slowing down region (ev to Kev) and the thermal region (around 0.025 ev).

For a given energy region – thermal, epithermal, resonance, fast – not all the possible reactions are equally important. Which reaction is important depends on the target nucleus and the neutron energy. Generally speaking the important types of interactions, in the order of increasing complexity from the standpoint of theoretical understanding, are:

(n,n) – elastic scattering. There are two processes, potential scattering which is neutron interaction at the surface of the nucleus (no penetration) as in a billiard ball-like collision, and resonance scattering which involves the formation and decay of a compound nucleus.

(n, γ) -- radiative capture.

(n,n') -- inelastic scattering. This reaction involves the excitation of nuclear levels.

(n,p), (n, α), … -- charged particle emission.

(n,f) -- fission.

If we were interested in fission reactors, the reactions in the order of importance would be fission, capture (in fuel and other reactor materials), scattering (elastic and inelastic), and fission product decay by β-emission. In this lecture we will mostly study elastic (or potential) scattering. The other reactions all involve compound nucleus formation, a process we will discuss briefly around the end of the semester.

The Q-Equation

Consider the reaction, sketched in Fig. 15.1, where an incoming particle, labeled 1, collides with a target nucleus (2), resulting in the emission of an outgoing particle (3), with the residual nucleus (4) recoiling. For simplicity we assume the target nucleus to be

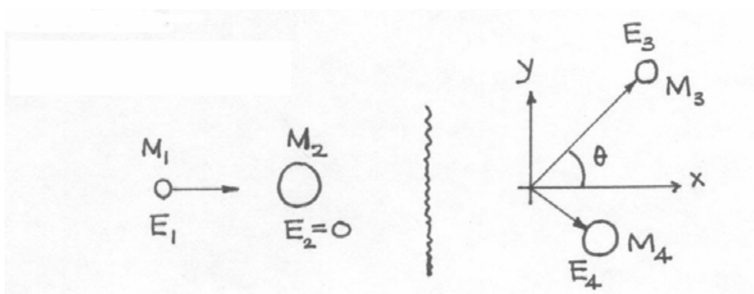

Fig. 15.1. A two-body collision between incident particle 1 and target particle 2, which is at rest, leading to the emission of particle 3 at an angle θ and a recoiling residual particle 4.

at rest, $E_2 = 0$. This is often a good approximation because for a target at room temperature E_2 is 0.025 ev; except for incoming neutrons in the thermal energy region, E_1 typically will be much greater than E_2. We will derive an equation relating the outgoing energy E_3 to the outgoing angle θ using the conservation of total energy and linear momentum, and non-relativistic kinematics,

$$(E_1 + M_1c^2) + M_2c^2 \quad (E_3 + M_3c^2) + (E_4 + M_4c^2) \tag{15.1}$$

$$\underline{p}_1 \quad \underline{p}_3 + \underline{p}_4 \tag{15.2}$$

Rewriting the momentum equation as

$$p_4^2 \quad (\underline{p}_1 - \underline{p}_3)^2$$

$$= \; p_1^2 + p_3^2 - 2p_1p_3\cos\theta \quad 2M_4E_4 \tag{15.3}$$

and recalling

$$Q \quad (M_1 + M_2 - M_3 - M_4)c^2$$

$$E_3 + E_4 - E_1 \tag{15.4}$$

we obtain

$$Q \quad E_3\left(1 + \frac{M_3}{M_4}\right) - E_1\left(1 - \frac{M_1}{M_4}\right) - \frac{2}{M_4}\sqrt{M_1M_3E_1E_3}\cos\theta \tag{15.5}$$

which is known as the Q-equation. Notice that the energies E_i and angle θ are in the laboratory coordinate system (LCS), while Q is independent of coordinate system (since Q can be expressed in terms of masses which of course do not depend on coordinate system). A typical situation is when the incident energy E_1, the masses (and therefore Q-value) are all known, and one is interested in solving (15.5) for E_3 in terms of $\cos\theta$, or vice versa.

Eq. (15.5) is actually not an equation for determining the Q-value; this is already known because all four particles in the reaction and their rest masses are prescribed beforehand. What then is the quantity that one can solve (15.5) to obtain? We should think of the Q-equation as a relation connecting the 12 degrees of freedom in any two-body collision problem, where two particles collide (as reactants) to give rise to two other particles (as products). The problem is completely specified when the velocities of the fours particles, or 12 degrees of freedom (each velocity has 3 degrees of freedom), are determined. Clearly not every single degree of freedom is an unknown in the situations of interest to us. Suppose we enumerate all the degrees of freedom to see which is given (known) and which is a variable. First if the direction of travel and energy of the incoming particle are given, usually the case, this specifies 3 degrees of freedom. Secondly it is customary to take the target nucleus to be stationary, so another 3 degrees of freedom are specified. Since conservations of energy and momentum must hold in any collision (three conditions since momentum and energy are related), this leaves three degrees of freedom that are not specified in the problem. If we further assume the emission of the outgoing particle (particle 3) is azimuthally symmetric (that is, emission is equally probably into a cone subtended by the angle θ), then only two degree of freedom are left. This way of counting shows that the outcome of the collision is completely determined if we just specify another degree of freedom. What variable should we take? Because we are often interested in knowing the energy or direction of travel of the outgoing particle, we can choose this last variable to be either E_3 or the scattering angle θ. In other words, if we know either E_3 or θ, then everything else (energy and direction) about the collision is determined. Keeping this in mind, it should come as no surprise that what we will do with (15.5) is to turn it into a relation between E_3 and θ.

Thus far we have used non-relativistic expressions for the kinematics. To turn (15.5) into the relativistic Q-equation we can simply replace the rest mass M_i by an effective mass, $M_i^{eff} \quad M_i + T_i / 2c^2$, and use the expression $p^2 \quad 2MT + T^2 / c^2$ instead of $p^2 \quad 2ME$. For photons, we take $M^{eff} \quad h\nu / 2c^2$.

Inspection of (15.5) shows that it is a quadratic equation in the variable x = $\sqrt{E_3}$. An equation of the form $ax^2 + bx + c \quad 0$ has two roots,

$$x_{\pm} \quad \left[-b \pm \sqrt{b^2 \, 4ac}\right] / 2a \tag{15.6}$$

which means there are in general two possible solutions to the Q-equation, $\pm\sqrt{E_3}$. For a solution to be physically acceptable, it must be real and positive. Thus there are situations where the Q-equation gives one, two, or no physical solutions [cf. Evans, pp. 413-415, Meyerhof, p. 178]. For our purposes we will focus on neutron collisions, in particular the case of elastic (Q = 0) and inelastic (Q < 0) neutron scattering. We will examine these two processes briefly and then return to a more detailed discussion of elastic scattering in the laboratory and center-of-mass coordinate systems.

Elastic vs. Inelastic Scattering

Elastic scattering is the simplest process in neutron interactions; it can be analyzed in complete detail. This is also an important process because it is the primary mechanism by which neutrons lose energy in a reactor, from the instant they are emitted as fast neutrons in a fission event to when they appear as thermal neutrons. In this case, there is no excitation of the nucleus, Q = 0; whatever energy is lost by the neutron is gained by the recoiling target nucleus. Let $M_1 = M_3 = m$ (M_n), and $M_2 = M_4 = M = Am$. Then (15.5) becomes

$$E_3\left(1+\frac{1}{A}\right) - E_1\left(1-\frac{1}{A}\right) - \frac{2}{A}\sqrt{E_1 E_3}\cos\theta \quad 0 \tag{15.7}$$

Suppose we ask under what condition is $E_3 = E_1$? We see that this can occur only when $\theta = 0$, corresponding to forward scattering (no interaction). For all finite θ, E_3 has to be less than E_1, which is reasonable because some energy has to be given to the energy of recoil, E_4. One can show that the maximum energy loss by the neutron occurs at $\theta \quad \pi$, which corresponds to backward scattering,

$$E_3 = \alpha E_1, \qquad \alpha \quad \left(\frac{A-1}{A+1}\right)^2 \tag{15.8}$$

Eq.(15.7) is the starting point for the analysis of neutron moderation (slowing down) in a scattering medium. We will return to it later in this lecture.

Inelastic scattering is the process by which the incoming neutron excites the target nucleus so it leaves the ground state and goes to an excited state at an energy E* above the ground state. Thus Q = -E* (E* > 0). We again let the neutron mass be m and the target nucleus mass be M (ground state) or M* (excited state), with $M^* = M + E^*/c^2$. Since this is a reaction with negative Q, it is an endothermic process requiring energy to be supplied before the reaction can take place. In the case of scattering the only way energy can be supplied is through the kinetic energy of the incoming particle (neutron). Suppose we ask what is minimum energy required for the reaction, the *threshold* energy? To find this, we look at the situation where no energy is given to the outgoing particle, E_3 ~0 and $\theta \sim 0$. Then (15.5) gives

$$-E^* = -E_{th}\left(\frac{M_4 - M_1}{M_4}\right), \qquad \text{or } E_{th} \sim E^*(1+1/A) \tag{15.9}$$

where we have denoted the minimum value of E_1 as E_{th}. Thus we see the minimum kinetic energy required for reaction is always greater than the excitation energy of the nucleus. Where does the difference between E_{th} and E* go? The answer is that it goes into the center-of-mass energy, the fraction of the kinetic energy of the incoming neutron (in the laboratory coordinate) that is not available for reaction.

Relations between Outgoing Energy and Scattering Angle

We return to the Q-equation for elastic scattering to obtain a relation between the energy of the outgoing neutron, E_3, and the angle of scattering, θ. Again regarding (15.5) as a quadratic equation for the variable $\sqrt{E_3}$, we have

$$E_3 - \frac{2}{A+1}\sqrt{E_1 E_3}\cos\theta - \frac{A-1}{A+1}E_1 \quad 0 \tag{15.10}$$

with solution in the form,

$$\sqrt{E_3} \qquad \frac{1}{A+1}\sqrt{E_1}\left(\cos\theta + \left[A^2 - \sin^2\theta\right]^{1/2}\right) \tag{15.11}$$

This is a perfectly good relation between E_3 and θ (with E_1 fixed), although it is not a simple one. Nonetheless, it shows a one-to-one correspondence between these two variables. This is what we meant when we said that the problem is reduced to only degree of freedom. Whenever we are given either E_3 or θ we can immediately determine the other variable. The reason we said that (15.11) is not a simple relation is that we can obtain another relation between energy and scattering angle, except in this case the scattering angle is the angle in the center-of-mass coordinate system (CMCS), whereas θ in (15.11) is the scattering angle in the laboratory coordinate system (LCS). To find this simpler relation we first review the connection the two coordinate systems.

Relation between LCS and CMCS

Suppose we start with the velocities of the incoming neutron and target nucleus, and those of the outgoing neutron and recoiling nucleus as shown in the Fig. 15.2.

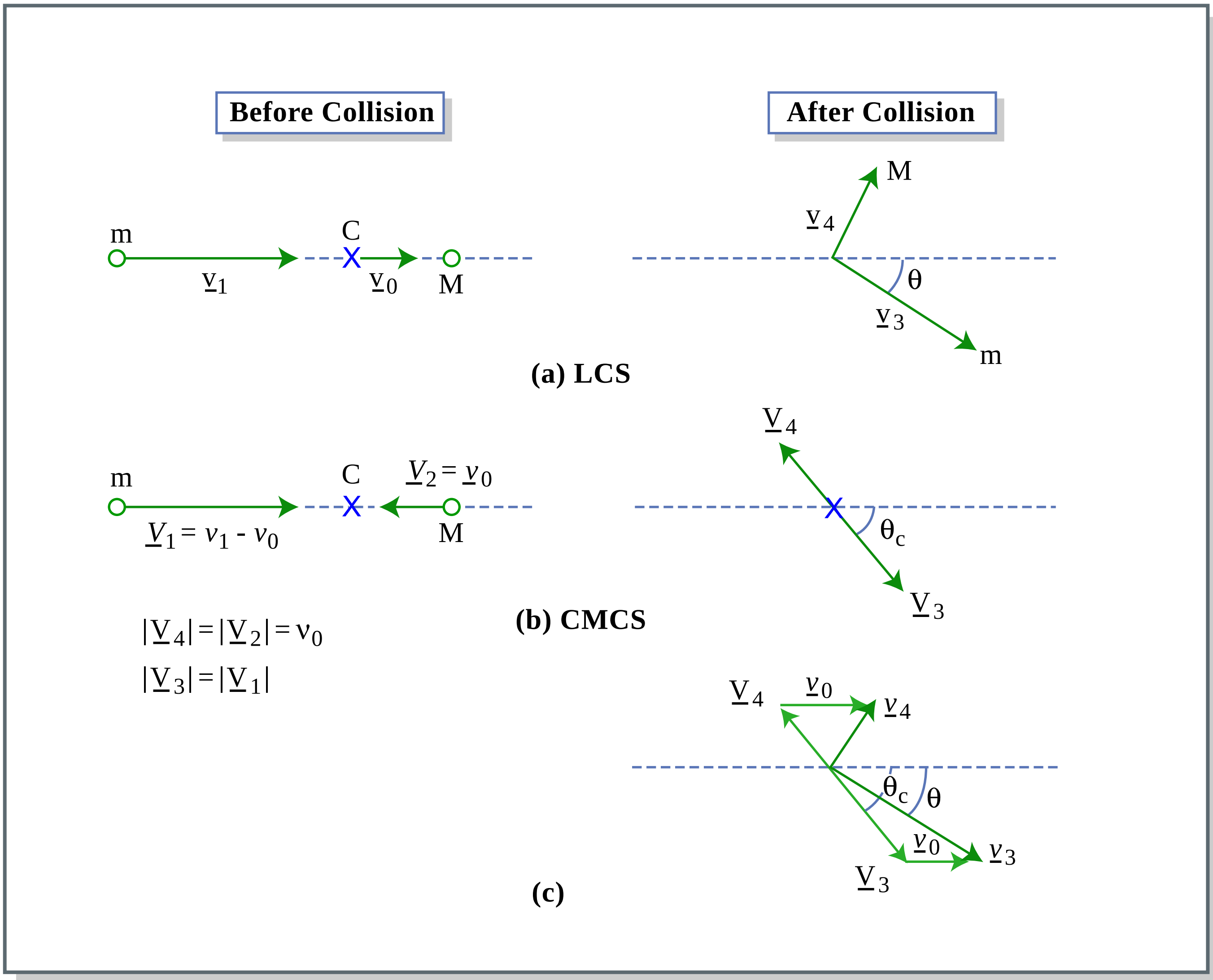

Fig. 15.2. Elastic scattering in LCS (a) and CMCS (b), and the geometric relation between LCS and CMCS post-collision velocity vectors (c).

Figure by MIT OCW.

In this diagram we denote the LCS and CMCS velocities by lower and upper cases respectively, so $\underline{V}_i = \underline{v}_i - \underline{v}_o$, where $\underline{v}_o = [1/(A+1)]\underline{v}_1$ is the velocity of the center-of-mass. Notice that the scattering angle in CMCS is labeled as θ_c. We see that in LCS the center-of-mass moves in the direction of the incoming neutron (with the target nucleus at rest), whereas in CMCS the target nucleus moves toward the center-of-mass which is stationary by definition. One can show (in a problem set) that in CMCS the post-collision velocities have the same magnitude as the pre-collision velocities, the only effect of the collision being a rotation, from $\underline{V}_1$ to $\underline{V}_3$, and $\underline{V}_2$ to $\underline{V}_4$.

Part (c) of Fig. 15.1 is particularly useful for deriving the relations between LCS and CMCS velocities and angles. Perhaps the most important relation is that between the outgoing speed v_3 and the scattering angle in CMCS , θ_c. We can write

$$\frac{1}{2}mv_3^2 \quad \frac{1}{2}m\left(\underline{V}_3 + \underline{v}_o\right)^2$$

$$= \frac{1}{2}m\left(V_3^2 + v_o^2 + 2V_3 v_o \cos\theta_c\right) \tag{15.12}$$

or

$$E_3 \quad \frac{1}{2}E_1\left[(1+\alpha) + (1-\alpha)\cos\theta_c\right] \tag{15.13}$$

where $\alpha \quad \left[(A-1)/(A+1)\right]^2$. Compared to (15.11), (15.13) is clearly simpler to manipulate. These two relations must be equivalent since no approximations have been made in either derivation. Taking the square of (15.11) gives

$$E_3 \quad \frac{1}{(A+1)^2}E_1\left(\cos^2\theta + A^2 - \sin^2\theta + 2\cos\theta\left[A^2 - \sin^2\theta\right]^{1/2}\right) \tag{15.14}$$

To demonstrate the equivalence of (15.13) and (15.14) one needs a relation between the two scattering angles, θ and θ_c. This can be obtained from Fig. 15.1(c) by writing

$$\cos\theta \quad \left(v_o + V_3\cos\theta_c\right)/v_3$$

$$= \frac{1 + A\cos\theta_c}{\sqrt{A^2 + 1 + 2A\cos\theta_c}} \tag{15.15}$$

The relations (15.13), (15.14), and (15.15) all demonstrate a one-to-one correspondence between energy and angle or angle and angle. They can be used to transform distributions from one variable to another, as we will demonstrate in the discussion of energy and angular distribution of elastically scattered neutrons in the following lecture.