22.01 Fall 2015, Problem Set 6 (Normal Version Solutions)

Due: November 2, 11:59PM on Stellar

November 14, 2015

Complete all the assigned problems, and do make sure to show your intermediate work. Please upload your full problem set in PDF form on the Stellar site. Make sure to upload your work at least 15 minutes early, to account for computer/network issues.

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1 Conceptual Questions

1. Explain, using stopping power expressions and cross sections, why the energy loss due to ionization drops off so sharply with increasing energy, while radiation loss increases linearly.
The complete form of the non-relativistic stopping power expression for any charged particle is as follows:

$$-\frac{dT}{dx} = \frac{4\pi N Z_1^2 Z_2 e_c^4}{m_e v^2} ln\left(\frac{2m_e v^2}{\bar{I}}\right) \tag{1}$$

Neglecting the constant terms, this takes the following energy-dependent form:

$$-\frac{dT}{dx} \propto \frac{1}{E} ln\left(E\right) \tag{2}$$

This implies that at lower energies, the $\frac{1}{E}$ term completely dominates, and it drops off very sharply with increasing energy. This is solely due to the charged particle spending less time near each given electron, because it is moving faster. This also relates directly to the differential cross section for energy as follows, from Yip Equation 11.15:

$$-\frac{dT}{dx} = N \int_0^T E \frac{d\sigma}{dE} dE \tag{3}$$

At much higher energies, the ability of a particle with a given charge to ionize electrons farther and farther away (at higher impact parameters) increases, but not that quickly.

2. Explain the quantitative differences in stopping power of electrons as they reach relativistic speeds. What energy cutoff do you consider relativistic, and why?
As particles reach relativistic speeds, their effective mass increases, meaning that they will be deflected less per unit length for the same applied Coulomb force between them and the electrons in the medium. This gives them greatly increased stopping power. Let's say that we choose an energy cutoff where the stopping power is increased by no more than 5% at its peak from its non-relativistic value. Then we can graph the following ratio as a function of β:

$$Ratio = 1.05 = \frac{\cancel{\frac{4\pi N Z_1^2 Z_2 e_c^4}{m_e v^2}} ln\left(\frac{2m_e v^2}{\bar{I}^2(1-\beta^2)} - \beta^2\right)}{\cancel{\frac{4\pi N Z_1^2 Z_2 e_c^4}{m_e v^2}} ln\left(\frac{2m_e v^2}{\bar{I}}\right)} \tag{4}$$

where we have used the relativistic stopping power expression from Yip, Equation 11.12. Solving for β:

$$1.05 = \frac{ln\left(\frac{2m_e v^2}{\bar{I}^2(1-\beta^2)} - \beta^2\right)}{ln\left(\frac{2m_e v^2}{\bar{I}}\right)} \tag{5}$$

$$1.05 = ln\left(\frac{2m_e v^2}{\bar{I}^2\left(1-\beta^2\right)} - \beta^2 - \frac{2m_e v^2}{\bar{I}}\right) \tag{6}$$

$$e^{1.05} = \frac{2m_e v^2}{\bar{I}^2\left(1-\beta^2\right)} - \beta^2 - \frac{2m_e v^2}{\bar{I}} \tag{7}$$

and substituting in $v = \beta c$:

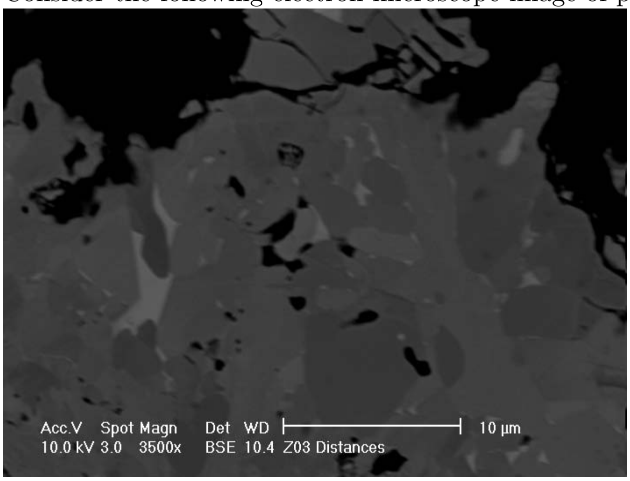
$$0 = \frac{2m_e \beta^2 c^2}{\bar{I}^2\left(1-\beta^2\right)} - \beta^2 - \frac{2m_e \beta^2 c^2}{\bar{I}} - e^{1.05} \tag{8}$$

$$0 = \beta^2\left[\frac{2m_e c^2}{\bar{I}}\left[\frac{1}{\bar{I}\left(1-\beta^2\right)} - 1\right] - 1\right] - e^{1.05} \tag{9}$$

$$0 = \beta^2\left[\frac{1.022\,MeV}{\bar{I}}\left[\frac{1}{\bar{I}\left(1-\beta^2\right)} - 1\right] - 1\right] - e^{1.05} \tag{10}$$

This function was graphed as a function of β, and its intersection with the x-axis will reveal at what value of β the stopping power increases by 5% at its peak. This value was found to be $\beta \approx 0.27$, even over a wide range of $\bar{I}$ ranging from 5-50keV.

3. Consider the following electron microscope image of palladium diffusion into zirconium carbide:



Where is the palladium in this image, and how do you know, based on your knowledge of electron interaction mechanisms with matter? Back up your answer with a relevant quantitative estimate of electron interactions. Using an image processing program, measure the relative brightnesses of various types of spots in the images. Can you guess the average atomic number of each of the spots? In other words, is brightness linearly proportional to the type of electron interaction(s) that you are interested in?

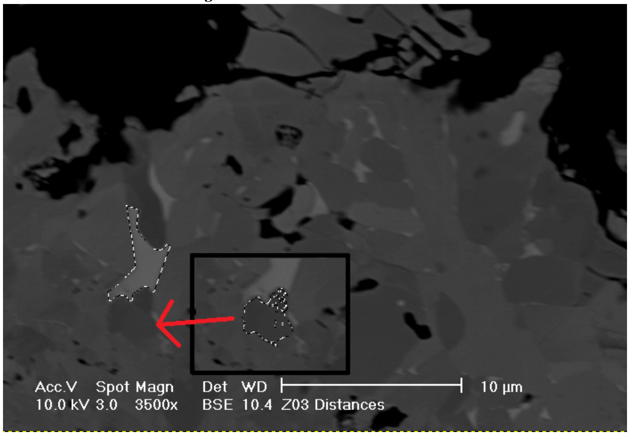
The palldium is the white spots in the image. This was an image made with the BSE (Back-Scattered Electron) detector, where electrons fired from the microscope scatter directly backwards into a detector. The cross section for backscattering is as follows:

$$\sigma_{bs} = \frac{\alpha Z^{2\,4}}{4\beta} \tag{11}$$

where $\alpha = 1\,barn$, and β is defined as above.

Using the GIMP image manipulation program, regions were automatically selected using the "Fuzzy

Select Tool" to represent the brightness of bright (Pd) and dark (ZrC) regions. These regions are shown in the image below:



Using the Colors→Info→Histogram menu, the average 8-bit brightnesses of these two regions were found to be 85.7 (bright) and 53.0 (dark), respectively. Using $\mathrm{Z_{Pd}} = 46$ *and* $\bar{Z}_{ZrC} = 23$*, this would yield a ratio of exactly* $\frac{\sigma_{Pd}}{\sigma_{ZrC}} = 4$*. The discrepancy between the expected brightness ratio and the observed one is that in an electron microscope, the user can dynamically change the brightness (offset) and contrast (multiplying factor) of the luminosity of the entire image at once. This was done in this case to enhance contrast in the image, making a direct cross section comparison impossible using just image analysis.*

2 Applied Questions

In these questions, consider the differences between x-ray and proton cancer therapy

1. Explain, using attentuation and stopping power, why protons are far more effective at damaging a localized tumor. Draw any applicable range relations and/or attenuation graphs to make your point. *The stopping power of protons vs. heavy ions in matter is much lower, owing to the fact that a mass ratio must be added to the electron stopping power expression in Equation 1 to yield:*

$$-\frac{dT}{dx} = \frac{4\pi N Z_1^2 Z_2 e_c^4}{m_e v^2} \frac{M}{m_e} ln\left(\frac{2m_e v^2}{\bar{I}}\right) \tag{12}$$

where M is the mass of the ion. If the proton has a lower stopping power, then it moves farther through materials, including human pieces & parts, to reach the tumor. Plus, once the protons reach the tumor, the width of the Bragg peak is narrower, leading to more damage to the tumor and less to the surrounding tissue. A SRIM simulation of 10MeV protons and 700MeV aluminum ions, tuned to reach the same distance, was performed as a visual comparison. We assumed that the particles would have to travel through 1.2mm of water to reach their target. The sheer magnitude of the required acceleration for heavier ions shows how much more difficult it is to get into a tumor of any depth. The range relation important to note here is that range goes way down with increasing ion mass, all other things held constant. For a fixed depth, heavy ions are much closer to their Bragg peak than protons, so the amount of spread in the ionizations/damage they will produce is larger, delocalizing the damage to the tumor.

2. One way of ensuring a uniform dose to a tumor of finite size (not small) is called intensity modulated radiation therapy (IMRT), where the proton beam is modulated in energy and/or angle to shift the Bragg peak to different specific locations. The goal is to maximize dose to the whole tumor, while minimizing the dose to surrounding tissue. For the following questions, assume we are trying to treat a tumor 1cm in diameter, surrounded by 5cm of healthy tissue.

   (a) Derive a relationship between the required energy and the atomic mass of a singly charged ion required to reach the center of the tumor. This will tell you how big of an accelerator one needs to use each type of ion.

The range of a charged particle can be directly related to its expression for stopping power:

$$R = \int_0^{E_i} \left(-\frac{dE}{dx}\right)^{-1} dE \tag{13}$$

and we can use the normal expression for stopping power of a singly charged ion ($Z_1 = 1$) *from the Bethe-Bloch formula (which will reduce to something quite familiar:*

$$-\frac{dT}{dx} = \frac{4\pi N Z_2 e_c^4}{E_i} \frac{M}{m_e} ln\left(\frac{\gamma_e E_i}{\bar{I}}\right) \tag{14}$$

where $\gamma_e = \frac{4m_e M}{(M+m_e)^2} \approx \frac{4m_e M}{M^2} = \frac{4m_e}{M}$. *Furthermore, let's assume that everything squishy is basically water, with a* $\bar{Z}_{H_2O} = 3.33$ *and a number density at room temperature of* $3.33 \cdot 10^{28} \frac{molecules}{m^3} = 1 \cdot 10^{29}$ *We'll also assume that* $M \approx A$ *for the mass of the ion, and that* $\bar{I}_{H_2O}$=*12.6* eV, *and* $m_e = 0.00055$ *amu, and in the non-relativistic case* $E_i = \frac{1}{2}Mv^2$. *This gives us* $\gamma_e = \frac{4(0.00055)A}{(A+0.00055)^2} \approx \frac{4(0.00055)A}{A^2} = \frac{0.0022}{A}$, *and the following stopping power expression:*

$$-\frac{dT}{dx} = \frac{4\pi N Z_2 e_c^4}{\frac{1}{2}Mv^2} \frac{M}{m_e} ln\left(\frac{\frac{4m_e}{M}\frac{1}{2}Mv^2}{\bar{I}}\right) = \frac{8\pi N Z_2 e_c^4}{m_e v^2} ln\left(\frac{2m_e v^2}{\bar{I}}\right) \tag{15}$$

That looks quite familiar! Because it will be easier in the integral, let's use the Bethe-Bloch formula (Equation 14) as-is:

$$R = \int_0^{E_i} \left(\frac{4\pi N Z_2 e_c^4}{E} \frac{A}{0.00055} ln\left(\frac{0.0022E}{12.6A}\right)\right)^{-1} dE = \int_0^{E_i} \frac{0.00055E}{4\pi A N Z_2 e_c^4 ln\left(\frac{0.000175E}{A}\right)} dE = 0.055\,m \tag{16}$$

$$R = \frac{0.00055}{4\pi A N Z_2 e_c^4} \int_0^{E_i} \frac{E}{ln\left(\frac{0.000175E}{A}\right)} dE = 0.055\,m \tag{17}$$

This integral will explode, because it's undefined at E=0. Therefore, we can change the lower integrand to $\bar{I}$, or the minimum energy required to produce even one ionization, because below that we don't get anymore cell damage by ionization:

$$0.055\,m = \frac{0.00055}{4\pi A N Z_2 e_c^4} \int_{\bar{I}}^{E_i} \frac{E}{ln\left(\frac{0.000175E}{A}\right)} dE \tag{18}$$

The integral of what's left inside is described by the "Exponential Integral" function (Ei), which is defined as follows[1]:

$$Ei\,(n, T) = \int_1^{\infty} \frac{e^{-Tx}}{x^n} dx \tag{19}$$

Using Maple on Athena to integrate the function inside the integral:

$$\int_{\bar{I}}^{E_i} \frac{E}{ln\left(\frac{0.000175E}{A}\right)} dE = \frac{A^2}{0.000175^2}\left[Ei\left(1, -2\,ln\left(\frac{0.000175E_i}{A}\right)\right) + \cancelto{0}{Ei\left(1, -2\,ln\left(\frac{0.000175\bar{I}}{A}\right)\right)}\right] \tag{20}$$

[1]See http://www.mathworks.com/help/symbolic/mupad_ref/ei.html for an explanation

Because the second term containing $\bar{I}$ is so much smaller than the first term containing E_i, we just forget about it. This reduces our expression to the following:

$$0.055\,m = \frac{0.00055}{4\pi A N Z_2 e_c^4}\frac{A^2}{0.000175^2}Ei\left(1, -2\,ln\left(\frac{0.000175E_i}{A}\right)\right) \tag{21}$$

$$0.055\,m = \frac{0.00055}{4\pi N Z_2 e_c^4}\frac{A}{0.000175^2}Ei\left(1, -2\,ln\left(\frac{0.000175E_i}{A}\right)\right) \tag{22}$$

$$0.000128\,m = \frac{A}{N e_c^4}Ei\left(1, -2\,ln\left(\frac{0.000175E_i}{A}\right)\right) \tag{23}$$

This, in principle, would yield a graphical function between A and E_i. Let's instead make a much simpler assumption, that because we are far in the low energy regime, we can use the range-energy relationship in Equation 11.25 of Yip:

$$R \propto \int_{\bar{I}}^{E_i} E dE \approx \frac{E_i^2}{2} \tag{24}$$

because the broad minimum in the stopping power formula is expected at around $\sim 3Mc^2$, which would be about 2800MeV for protons, and even higher for other ions. For reference, it takes about a 250MeV accelerator to get the protons anywhere in the human body. Using this range-energy relationship in Equation 18, we get:

$$0.055\,m = \frac{0.00055}{8\pi A N Z_2 e_c^4}E_i^2 \tag{25}$$

Rearranging to isolate E_i we get the following:

$$\sqrt{800\pi A N Z_2 e_c^4} = E_i \tag{26}$$

This means that the energy required simply increases with the square root of the ion's mass.

(b) Now derive a relationship between the amount of ionization of each of these ions at their starting energy and within the Bragg peak. This gives ratio of the amount of damage to the tumor compared to the surrounding tissue directly from the ions themselves.
Here, we note that the number of ionizations per unit length is given by the following formula:

$$i = \frac{1}{W}\left(-\frac{dT}{dx}\right) \tag{27}$$

where W is the energy it takes to create an ion pair. To get the relationship between the ions at their starting energy and their Bragg peak, we just choose an ion (let's make it a proton) and a starting energy (let's make it 50MeV):

$$Ratio = \frac{\frac{\cancel{4\pi N Z_2 e_c^4}}{E_i}\frac{\cancel{M}}{\cancel{m_e}}ln\left(\frac{\gamma_e E_i}{\bar{I}}\right)}{\frac{\cancel{4\pi N Z_2 e_c^4}}{500\bar{I}}\frac{\cancel{M}}{\cancel{m_e}}ln\left(\frac{\gamma_e 500\cancel{\bar{I}}}{\cancel{\bar{I}}}\right)} = \frac{500\bar{I}\,ln\left(\frac{\gamma_e E_i}{\bar{I}}\right)}{E_i\,ln\left(500\gamma_e\right)} \tag{28}$$

where we assume the protons start at 50MeV, and they reach their maximum stopping power in the Bragg peak at about $500\bar{I}$. Substituting $E_i = 50MeV$ and $\bar{I}_{H_2O}$=12.6 eV, we get:

$$Ratio = \frac{500\,(12.6)\;ln\left(\frac{\gamma_e\left(5\cdot 10^7\right)}{12.6}\right)}{\left(5\cdot 10^7\right)\,ln\left(500\gamma_e\right)} \tag{29}$$

For protons, $\gamma_e = \frac{4(0.00055)A}{(A+0.00055)^2} \approx \frac{4(0.00055)A}{A^2} = 0.0022$*:*

$$Ratio = \frac{6300\, ln\left(\frac{0.0022\left(5\cdot 10^7\right)}{12.6}\right)}{\left(5\cdot 10^7\right)\, ln\,(1.1)} = \frac{57170}{4765508} = 0.012 \tag{30}$$

The protons therefore only do about 1% of the damage to the surrounding tissue at the entry point comapred to inside the tumor, making them quite effective.

(c) Consider the cases of electrons, protons, carbon ions, and iron ions. Which type of ion is most suitable for use in IMRT, and why? Hint: Consider other mechanisms of ion energy loss in tissue, and compare how intense they would be in a relative sense.
Protons should be most suitable for IMRT. Electrons must be accelerated to extremely relativistic speeds, at which point radiative losses at every distance would become significant. Heavier ions won't reach relativistic speeds (at least not too relativistic), though the heavier the ion, the higher the acceleration must be to reach a deep tumor.

(d) Develop an expression or graph for the amount of time the beam needs to spend at each depth of the tumor to apply the same dose at every location. Assume a one-dimensional tumor.
In this case it's a matter of the number of ion pairs applied per unit time. Assuming that the Bragg peak is very narrow, each spot of irradiation (each dV differential volume element) sees the same, low-energy protons entering and stopping inside it. Therefore, the amount of time per spot is roughly equal.
To be more thorough, however, one must consider the excess damage done by protons at a shallower depth on their way to a deeper tumor. Then, we know that the amount of "beam-on" time for any given spot on the 1D tumor must increase with distance from the beam entry point. Let's assume that the relationship from part (b) holds true, and the proton beam does a fraction $\boldsymbol{f}$ *of the damage outside the Bragg peak compared to inside it, wherever it is. Then for a given tumor distance x and a Bragg peak thickness dx, each spot would require the beam-on time of the deepest spot, minus all the other exposures that it saw as the beam passed right through. The deepest spot would require a beam-on time of t. The second deepest spot would require a beam-on time of* $\boldsymbol{t}$*, minus* $\boldsymbol{ft}$ *accounting for the fraction* $\boldsymbol{f}$ *damage already done by the deeper spot. The third deepest spot would require a beam-on time of* $\boldsymbol{t}$*, minus* $\boldsymbol{ft}$ *from the deepest spot, minus the* $\boldsymbol{f}((1-f)\,t)$ *from the second deepest spot, and so on.*
Continuing with this pattern, each spot would require a time expressed as follows:

$$t_x = t \tag{31}$$

$$t_{x-dx} = t - ft = (1-f)\,t \tag{32}$$

$$t_{x-2dx} = t - ft - f((1-f)\,t) = (1-f)\,t - ft(1-f) = (1-f)\,t\,(1-f) = (1-f)^2\,t \tag{33}$$

$$t_{x-3dx} = t - ft - f((1-f)\,t) - f((1-f)^2\,t) = (1-f)\,t\,((1-f) - f\,(1-f)) = (1-f)^2\,t\,(1-f) = (1-f)^3\,t \tag{34}$$

$$t_{x-ndx} = (1-f)^n\,t \tag{35}$$