22.01 Fall 2015, Problem Set 2 Solutions

Due: September 28, 11:59PM on Stellar

September 27, 2015

Complete all the assigned problems, and do make sure to show your intermediate work. Please upload your full problem set in PDF form on the Stellar site. Make sure to upload your work at least 15 minutes early, to account for computer/network issues.

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1 Predicting Nuclear Stability

Using the Table of Nuclides, answer the following questions about sulfur (Z=16):

1. Using the excess mass from the table of nuclides, calculate the binding energy and the binding energy per nucleon of ^{32}S. Use the table of nuclides to check your answer.
   The excess mass, binding energy, mass number (A), and binding energy per nucleon are calculated as follows:

$$\Delta\,[amu] = A\,[amu] - M\,(A,Z)\,[amu]\,; \qquad M\,(A,Z) = A - \Delta \tag{1}$$

$$\Delta\,[MeV] = (A\cancel{[amu]} - M\,(A,Z)\,\cancel{[amu]})\left(\frac{931.49\,\frac{MeV}{\cancel{c^2}}}{1\,\cancel{amu}}\right)\cancel{c^2} \tag{2}$$

$$BE\,(A,Z)\,[amu] = (ZM_p + (A-Z)\,M_n - M\,(A,Z))\,[amu] \tag{3}$$

$$BE\,(A,Z)\,[MeV] = ((ZM_p + (A-Z)\,M_n - A + \Delta)\,[amu])\left(\frac{931.49\,\frac{MeV}{\cancel{c^2}}}{1\,\cancel{amu}}\right)\cancel{c^2} \tag{4}$$

$$M_p = 1.0073\,amu; \qquad M_n = 1.0087\,amu \tag{5}$$

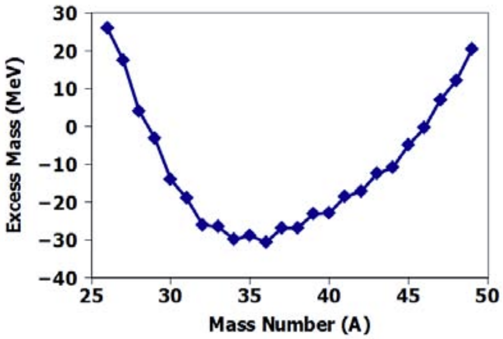
   Combining equations 4 and 5:

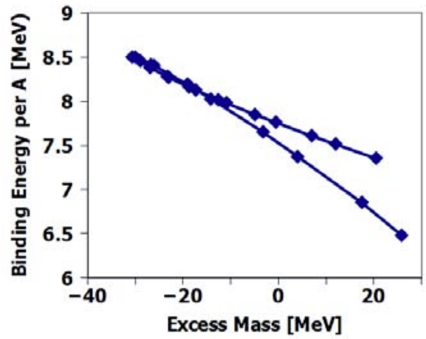
$$BE\,(A,Z)\,[MeV] = (((1.0073 - 1.0084)\,Z + (1.0084 - 1)\,A)\,[amu])\left(\frac{931.49\,\frac{MeV}{\cancel{c^2}}}{\cancel{amu}}\right)\cancel{c^2} + \Delta\,[MeV] \tag{6}$$

$$BE\,(A,Z)\,[MeV] = ((0.0084A - 0.0011Z)\,[amu])\,931.49\,MeV + \Delta\,[MeV] \tag{7}$$

   The data are shown on the attached spreadsheet for your comparison.

2. Graph the excess mass of each isotope of sulfur as a function of mass number (A), and as a function of binding energy per nucleon. What trends do you see?
   The two graphs, plotted separately, are shown here:





The first graph shows clear evidence of a mass parabola, in this case with a minimum excess mass (maximum stability) for a constant A. The second graph shows a very tight relation between excess mass and binding energy per nucleon. One important trend in the first graph is that even nuclei tend to have lower excess masses, and therefore higher stabilities, as would be expected from our liquid drop model of the nucleus. This effect is most pronounced for the more stable nuclei.

3. How do you explain the deviation in the smooth-ish curve for ^{33}S?
As explained in problem 1.2, the deviations in the smooth-ish curve at ^{33}S is because it's an even-odd nucleus, compared to the two even-even nuclei on either side of it.

4. For each region where an increase is seen from the most stable isotope(s) (left third, middle third, right third), briefly say why the nuclei in each region are most unstable. (Hint: What do you know about the relative number of protons and neutrons in a nucleus, and how does that help determine stability?)
On the left third of the first graph, nuclei are "proton-rich," meaning that the asymmetry reduction in binding energy is very strong. In addition, the Coulombic repulsion term, which remains constant at constant Z, plays a stronger role in reducing the A-dependent volumetric binding energy.
In the middle of the graph, it is the pairing term that seems to change the nuclear stability the most, as evidenced by the oscillatory motion in the curve.
On the right third of the first graph, the assymetry term dominates in reducing binding energy (increasing excess mass), as the nuclei are "neutron-rich."

2 Liquid-Drop Nuclear Models

For these questions, consider the liquid drop model of nuclear mass, which states that the mass of a nucleus can be empirically calculated as in Eq. 4.10 (p. 59) of *Nuclear Radiation Interactions* by S. Yip.

1. Explain the origin of each additive term in this expression. Pay particular attention to the exponents in each one, and explain why they are what they are.
The semi-empirical mass formula, expressed as the binding energy, is as follows:

$$BE\left(A,Z\right)=a_vA-a_sA^{\frac{2}{3}}-a_c\frac{Z\left(Z-1\right)}{A^{\frac{1}{3}}}-a_a\frac{\left(N-Z\right)^2}{A}+a_p\delta;\quad \delta=\left\{\begin{matrix}\frac{1}{\sqrt{A}}\;even-even\\ 0\;even-odd\\ -\frac{1}{\sqrt{A}}\;odd-odd\end{matrix}\right\}\tag{8}$$

The origins of the five terms are as follows:
1) Volumetric binding energy, as the number of nuclear bonds increases proportionally to the number of nucleons
2) Surface energy reduction, because the nucleons on the free surface aren't bound to as many neighboring nuclei. The exponent comes from the relation between the formulas for the volume and surface area of a sphere.
3) Coulombic repulsive energy reduction, because all Z protons each feel the repulsive force by the other $(Z-1)$ protons. The repulsive forces are in one direction, or 1/3rd the

dimensionality of a sphere.
4) Assymetric energy reduction, stemming from a strong decrease in stability from unequal numbers of protons and neutrons. Nucleons have energy shell levels just like electrons in an atom, and they are most stable when corresponding shell levels are filled.
5) Pairing energy change, which increases stability for even-even nuclei (paired, filled energy levels), and decreases stability for odd-odd nuclei (more unpaired, unfilled energy levels.)

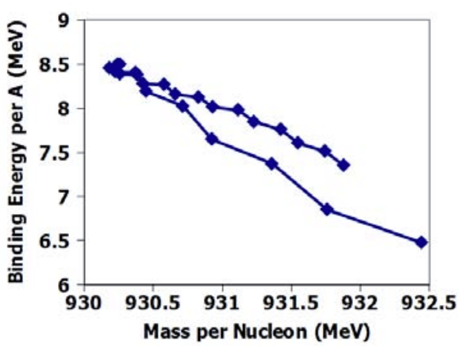
2. Why does the δ term in this expression change sign for odd/even nuclei?
   See (5) in the answer to problem 2.1

3. Modify equation 4.10 to empirically calculate the total rest mass of a given nucleus.
   The equation for the binding energy of a nucleus in terms of its rest mass (here Equation 3 in these solutions) can be combined with Equation 8 in these solutions to yield the following:

$$M(A,Z) = ZM_p + (A-Z)M_n - a_v A + a_s A^{\frac{2}{3}} + a_c \frac{Z(Z-1)}{A^{\frac{1}{3}}} + a_a \frac{(N-Z)^2}{A} - a_p \delta; \quad \delta = \begin{Bmatrix} \frac{1}{\sqrt{A}}\ even-even \\ 0\ even-odd \\ -\frac{1}{\sqrt{A}}\ odd-odd \end{Bmatrix} \tag{9}$$

   where all masses are expressed in equivalent MeV using the relation $\mathrm{E} = \mathrm{mc}^2$**.**

4. Graph the empirically calculated mass per nucleon for each isotope of sulfur, and compare it directly to your answer to problem 1.2. What differences do you observe?
   Here it is most instructive to directly compare the empirial liquid-drop model binding energy (Equation 9) to that calculated directly from the tabulated excess mass of each nucleus (Equation 7). The graph comparing the two is shown below:



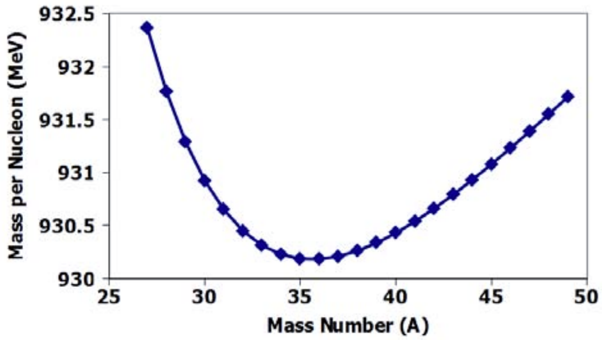
   There are two things that are evident from comparing this graph with the second one in problem 1.2:
   1) They are strikingly similar, showing a very similar trend between mass per binding energy and excess mass per binding energy.
   2) There are some deviations between the smooth, semi-empirical liquid-drop model of binding energy, and experimentally measured real binding energies. This means that there must be terms wihch the semi-empirical model does not fully capture.

5. Derive an expression for the most stable number of neutrons for a given nucleus with Z protons. Graph this expression as a function of Z. How does your prediction compare with the isotopes of sulfur?
   For this question, we can use Equation 9's full expression for the mass of a nucleus given A and Z. Then, assuming Z is held constant, we can take the derivative of M(A,Z) per nucleon (M/A) with respect to A and set it equal to zero to find the minimum mass. Taking a nucleus with a constant Z protons and a variable N neutrons would yield an always-increasing mass. Therefore, to make an equal comparison one must divide the mass by the total number of nucleons (A). Here we will forget about the pairing term,

as it's piecewise and doesn't have a smooth derivative:

$$\frac{M(A,Z)}{A} = \frac{Z}{A}M_p + \left(1 - \frac{Z}{A}\right)M_n - a_v + a_s A^{\frac{-1}{3}} + a_c\frac{Z(Z-1)}{A^{\frac{4}{3}}} + a_a\frac{(N-Z)^2}{A^2} \tag{10}$$

This function is graphed below:



The minimum of this graph appears to be at either A=35 or A=36, which agrees very well with the graph of excess mass showing the most stable nucleus at A=36. Looking at the KAERI table of nuclides, we see that this nucleus is indeed stable, though at a very small natural abundance. This is not due to nuclear stability, but rather due to the likelihood of fusion of its constituent parent nuclei.
The expression for the most stable number of nuclides can be found by setting the derivative $\frac{\partial\left(\frac{\mathrm{M}}{\mathrm{A}}\right)}{\partial \mathrm{A}} = 0$ **and solving for A:**

$$\frac{\partial\left(\frac{M}{A}\right)}{\partial A} = 0 = \frac{-Z}{A^2}M_p + \frac{Z}{A^2}M_n - \frac{a_s}{3}A^{\frac{-4}{3}} - \frac{4a_c}{3}Z(Z-1)A^{\frac{-7}{3}} + \frac{\partial}{\partial A}\left[a_a\frac{(A-2Z)^2}{A^2}\right] \tag{11}$$

Next we recognize that $(N-Z) = (A-2Z)$**:**

$$\frac{\partial\left(\frac{M}{A}\right)}{\partial A} = 0 = \frac{-Z}{A^2}M_p + \frac{Z}{A^2}M_n - \frac{a_s}{3}A^{\frac{-4}{3}} - \frac{4a_c}{3}Z(Z-1)A^{\frac{-7}{3}} + \frac{\partial}{\partial A}\left[a_a\frac{A^2 - 4AZ + 4Z^2}{A^2}\right] \tag{12}$$

$$\frac{\partial\left(\frac{M}{A}\right)}{\partial A} = 0 = \frac{-Z}{A^2}M_p + \frac{Z}{A^2}M_n - \frac{a_s}{3}A^{\frac{-4}{3}} - \frac{4a_c}{3}Z(Z-1)A^{\frac{-7}{3}} + 4Za_aA^{-2} - 8Z^2a_aA^{-3} \tag{13}$$

Next we recognize that $(M_n - M_p)$ **is approximately zero:**

$$0 = -\frac{a_s}{3}A^{\frac{-4}{3}} - \frac{4a_c}{3}Z(Z-1)A^{\frac{-7}{3}} + 4Za_aA^{-2} - 8Z^2a_aA^{-3} \tag{14}$$

Taking Z=16 for sulfur, and substituting the semi-empirical constants from the reading, the expression is as follows:

$$0 = -6A^{\frac{-4}{3}} - 230.4A^{\frac{7}{3}} + 1504A^{-2} - 48128A^{-3}; \quad A_{min\,M} = 35.15 \tag{15}$$

This is remarkably close to our estimate of A=36.

3 Q-Values and Nuclear Power

For these questions, consider equations 4.2 - 4.6 (pp. 54-55) in *Nuclear Radiation Interactions*.

1. Show that the Q-value for a reaction can be expressed solely in terms of nuclear binding energies (derive equation 4.6).
   The full Q-equation for a nuclear reaction of this general form:

$$i + I \rightarrow f + F + Q \tag{16}$$

can be expressed solely in terms of rest mass energies:

$$Q = (m_i + m_I - m_f - m_F)\, c^2 \tag{17}$$

and the equation defining the binding energy is the difference between the masses of a given nucleus and its constituent nucleons:

$$BE\,(A, Z) = ZM_p + NM_n - M\,(A, Z) \tag{18}$$

This can be rearranged to give the following:

$$M\,(A, Z) = ZM_p + NM_n - BE\,(A, Z) \tag{19}$$

Substituting this expression into Equation 17, we arrive at:

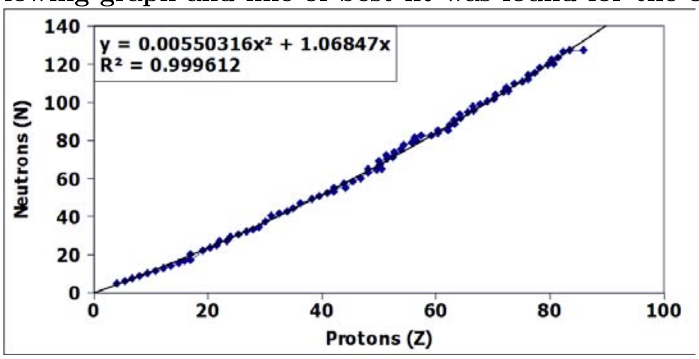
$$Q = ((Z_i + Z_I - Z_f - Z_F)\, M_p + (N_i + N_I - N_f - N_F)\, M_n - (BE_i\,(A, Z) + BE_I\,(A, Z) - BE_f\,(A, Z) - BE_F\,(A, Z))) \tag{20}$$

where in this equation, all quantities, including binding energies, are expressed in mass units. Because the total number of nucleons is unchanged in a nuclear reaction, the Z and N terms are either exactly or approximately conserved (the first case for simple rearrangement of nuclei, like alpha decay, the second case for things like beta decay)). That leaves only the binding energy terms, which when propagating the signs becomes:

$$Q = ((BE_f\,(A, Z) + BE_F\,(A, Z) - BE_i\,(A, Z) - BE_I\,(A, Z)))\, c^2 \tag{21}$$

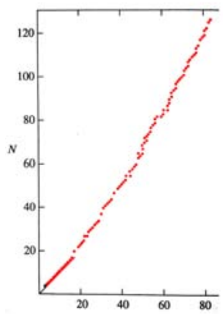
where in this last equation, the binding energies are expressed in MeV.

(a) Using a data extraction program like DataThief to get data points from Figure 4.4 (p. 58), produce an empirical expression for the optimum number of neutrons (N) for a given number of protons (Z) in a nucleus. Comment on the quality of that fit, and explain in which regions the fit is the best, and in which the fit is the worst.
Using the Web Plot Digitizer (http://arohatgi.info/WebPlotDigitizer/app/), the following graph and line of best fit was found for the odd nuclei:



The fit is extraordinarily god, except in regions around Z=32-35, 45-49, and 52-60. Still, for all the theoretical stuff, a simple polynomial fit does seem to work surprisingly well. The origin of this quadratic dependence likely lies in the Coulombic term, which slowly gets stronger with increasing nuclear size. It then requires more and more neutrons to screen neighboring neutrons.
Note that in order to most easily extract these points in an automated fashion, Figure 4.4 from Yip was photographed using a cell phone, and the points of the graph highlighted using GIMP (a free version of Photoshop-like software) to make them red, and more easily recognized. Remember, if you find yourself doing anything tedious (like looking for points one by one), look for another way! The graph used

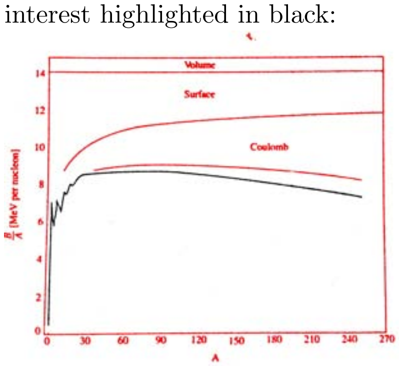
to produce this data looks as follows:



2. Using a data extraction program to get data from Figure 4.5 (p. 61) or a similar curve, produce a graph of the Q value for the fusion reaction of a given nucleus with a ^{4}He nucleus as a function of the nucleus' mass number (A):

$$M\,(A,\,Z) + ^4He \rightarrow M\,(A+4,\,Z+2) + Q \tag{22}$$

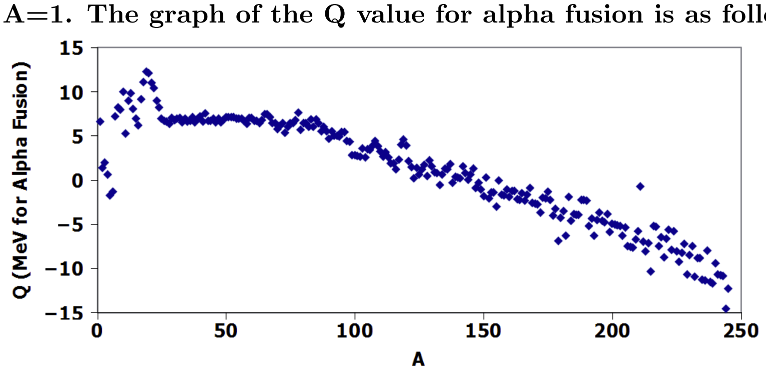
What noticeable features do you see? At which mass number (A) does the curve cross the x-axis? What does this say about the energetics of light vs. heavy nuclei for fusion (in other words, when is it exothermic, and when is it endothermic)?
The same data extraction program was used using the X-step algorithm, to obtain one data point per A value on the integers. The following image was used, taken from figure 4.5 in Yip, with the curve of interest highlighted in black:



The data extracted are in the attached BE-vs-A.csv file. Each point was analyzed as follows, taking a cue from Equation 21 in these solutions:

$$Q_{\alpha-Fusion} = A\left(\frac{BE}{A}\right)_A - (A-4)\left(\frac{BE}{A}\right)_{A-4} - 4\left(\frac{BE}{A}\right)_{A=4} \tag{23}$$

Values below A=4 had to be "cleaned," as duplicate values were found, and BE=0 when A=1. The graph of the Q value for alpha fusion is as follows:



This graph shows that it is actually energetically favorable in some cases for fusion of nuclei with helium, all the way up to about A=140. A quick sanity check shows that the binding energies of Cr-52 and He-4 are below those of Fe-56 by about 8 MeV, showing that 8 MeV would be released by fusing Cr-52 with He-4.