MAS.160 / MAS.510 / MAS.511 Signals, Systems and Information for Media Technology
Fall 2007

Causal FIR filter

Q:What is the definition of an FIR filter?

Causal FIR filter

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A: The output y at each sample n is a weighted sum of the present input, x[n], and past inputs, x[n-1], x[n-2],…, x[n-M].

Causal FIR filter

Q:What is the formula for an FIR filter?

Causal FIR filter

Q:What is the formula for an FIR filter?

$$y[n] = b_0 x[n] + b_1 x[n-1] + \ldots + b_M x[n-M]$$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

n	$x[n]$
-2	0
-1	0
0	1
1	0
2	0
3	0

$$x[n] = \delta[n] \longrightarrow$$

Causal FIR filter

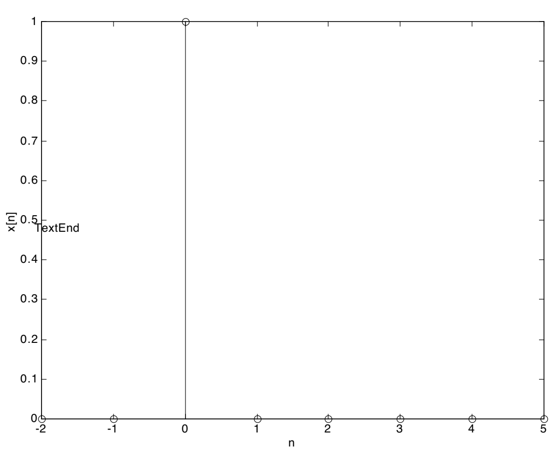
$$b_0 = 1, b_1 = 3, b_2 = 1$$

$$\longrightarrow y[n] = ?$$

$$y[n] = ?$$

$$h[n] = ?$$

$$y[1] = ?$$



$$\begin{bmatrix} n & x[n] \\ -2 & 0 \\ -1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 2 & 0 \\ 3 & 0 \end{bmatrix}$$

$x[n] = \delta[n]$ →

Causal FIR filter

$b_0 = 1, b_1 = 3, b_2 = 1$

→ $h[n] = y[n]\big|_{x[n]=\delta[n]}$

$$\begin{bmatrix} y[n] \\ 0 \\ 0 \\ b_0 \; 1 \\ b_1 \; 3 \\ b_2 \; 1 \\ 0 \end{bmatrix}$$

$$y[n] = 1x[n] + 3x[n-1] + 1x[n-2]$$

$$h[n] = y[n] = 1\delta[n] + 3\delta[n-1] + 1\delta[n-2]$$

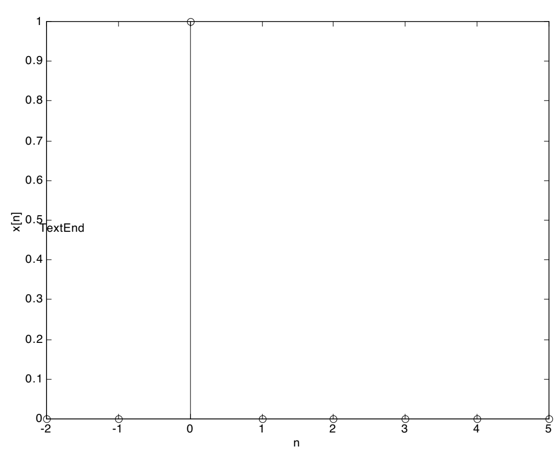
$$h[n] = b_n$$

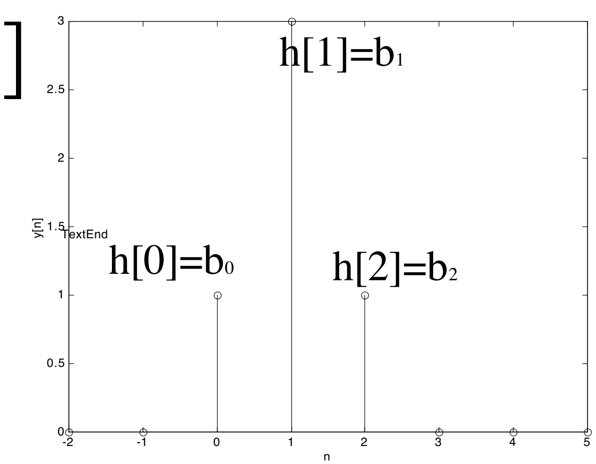
$$h[1] = 1\delta[1] + 3\delta[1-1] + 1\delta[1-2]$$

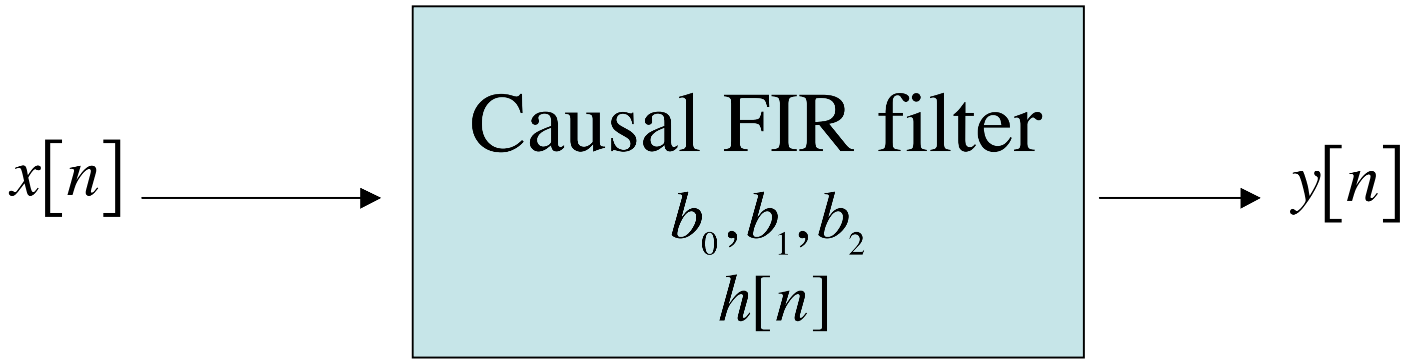
$$h[1] = 1\delta[1] + 3\delta[0] + 1\delta[-1]$$

$$h[1] = 1(0) + 3(1) + 1(0)$$

$$h[1] = 3$$







$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

weighted sum of delayed inputs

$$h[n] = b_n$$

$$y[n] = \sum_{k=0}^{M} h[k] x[n-k]$$

Convolution of impulse response and input

$$y[n] = h[n] * x[n]$$

$$\begin{bmatrix} n & x[n] \\ 0 & 0 \\ 1 & 0.88 \\ 2 & -0.84 \\ 3 & -0.06 \\ 4 & 0.90 \\ 5 & -0.81 \end{bmatrix}$$

$x[n] = \sin(2\pi \cdot 0.33n)u[n]$ →

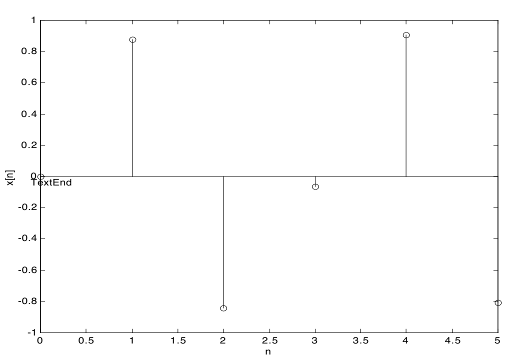
Causal FIR filter

$b_0 = 1, b_1 = 3, b_2 = 1$

$h[n] = [1, 3, 1]$

→ $y[n] = ?$

$y[3] = ?$



$$\begin{bmatrix} n & x[n] \\ 0 & 0 \\ 1 & 0.88 \\ 2 & -0.84 \\ 3 & -0.06 \\ 4 & 0.90 \\ 5 & -0.81 \end{bmatrix}$$

$x[n] = \sin(2\pi \cdot 0.33n)u[n]$ →

Causal FIR filter

$$b_0 = 1, b_1 = 3, b_2 = 1$$

$$h[n] = [1, 3, 1]$$

→

$$\begin{bmatrix} n & y[n] \\ 0 & 0 \\ 1 & 0.88 \\ 2 & 1.78 \\ 3 & -1.72 \\ 4 & -0.13 \\ 5 & 1.84 \end{bmatrix}$$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} h[k]x[n-k]$$

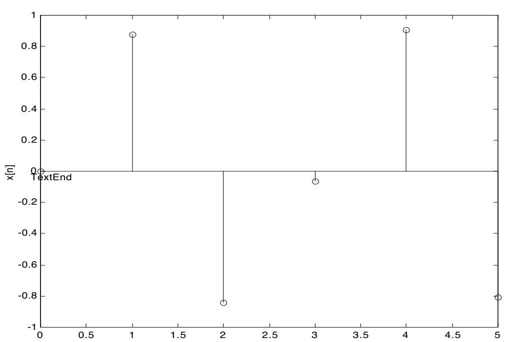
$$y[n] = 1x[n] + 3x[n-1] + 1x[n-2]$$

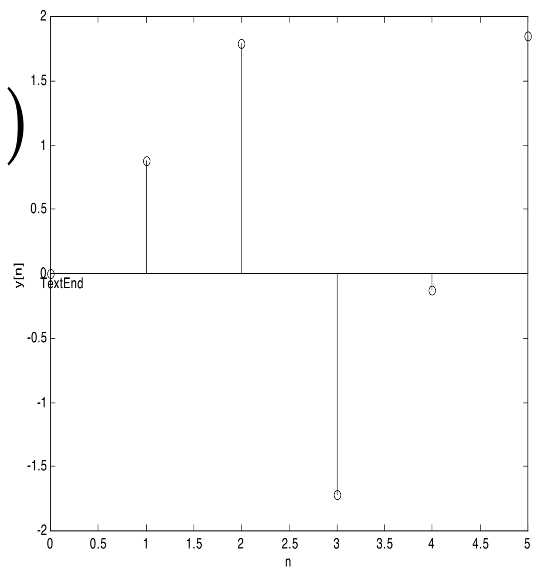
$$y[3] = 1x[3] + 3x[3-1] + 1x[3-2]$$

$$y[3] = 1x[3] + 3x[2] + 1x[1]$$

$$y[3] = 1(-0.06) + 3(-0.84) + 1(0.88)$$

$$y[3] = -1.72$$





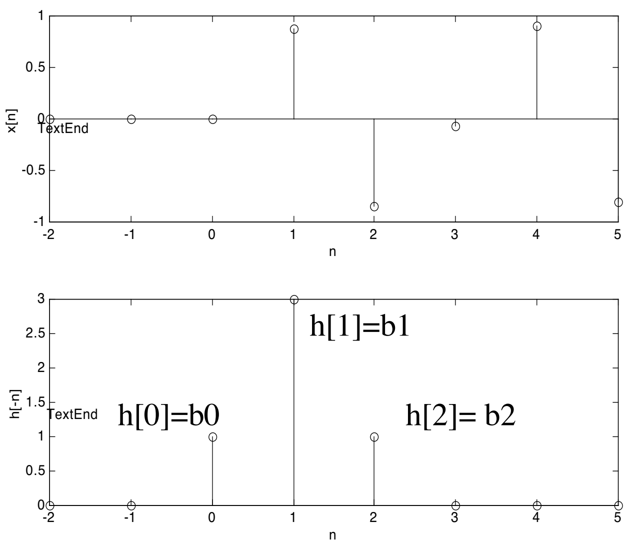
Graphical Convolution

$$y[n] = h[n] * x[n] = x[n] * h[n]$$

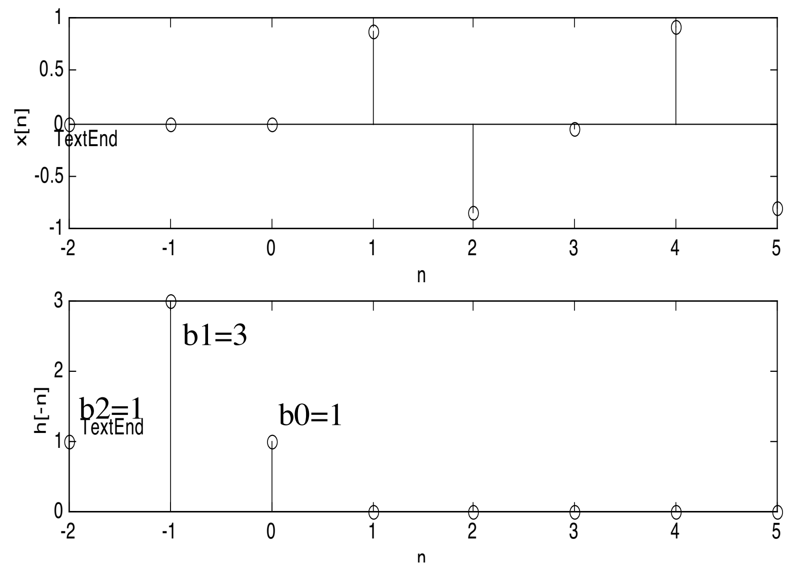
$$y[n] = \sum_{k=0}^{M} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

sum

multiply shift flip



flip



$$y[n] = x[n-2]h[n-(n-2)] + x[n-1]h[n-(n-1)] + x[n]h[n-n]$$

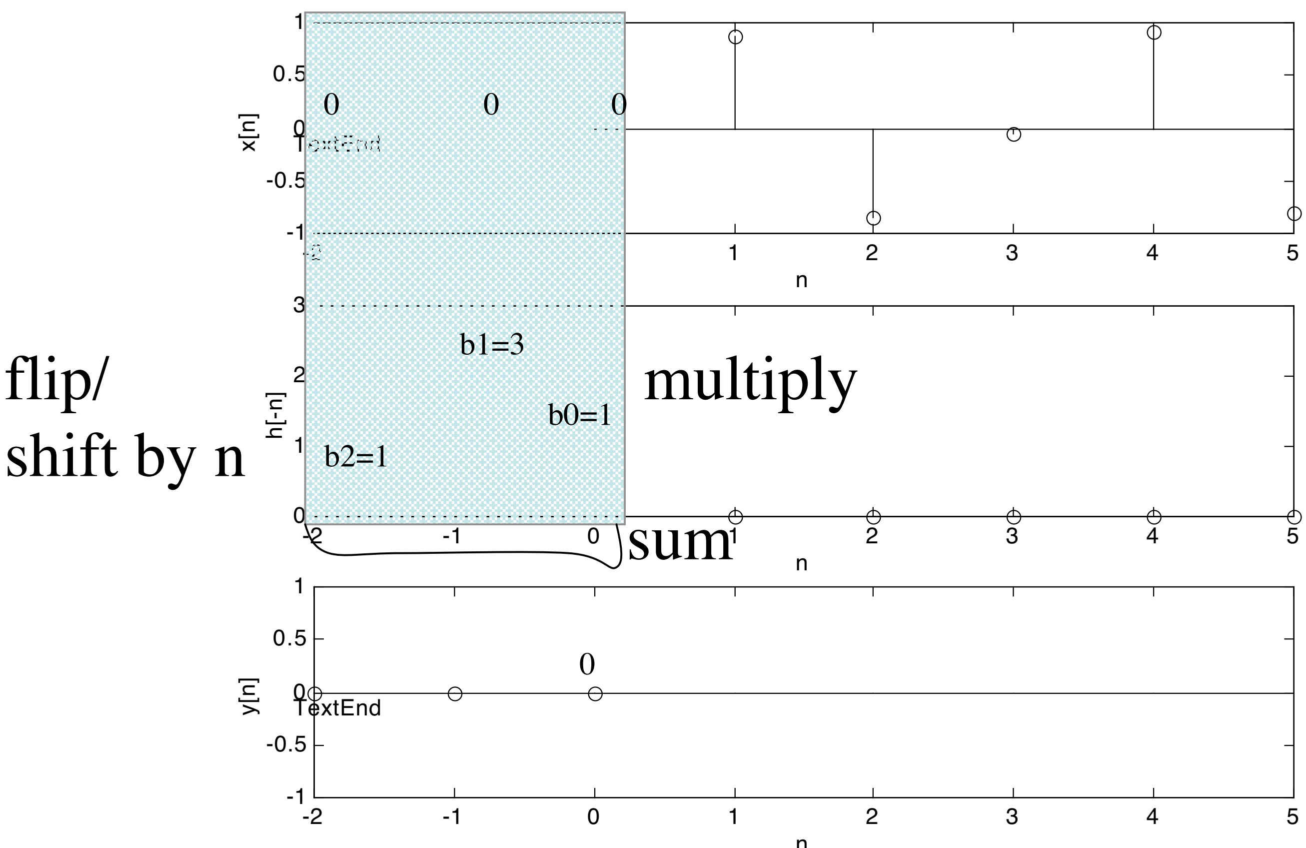
$$y[n] = h[0]x[n] + h[1]x[n-1] + h[2]x[n-2]$$

Graphical Convolution

$$y[n] = \sum_{k=-M}^{0} x[k]h[n-k]$$

sum, multiply, shift, flip

n=0



$$y[0] = x[-2]h[2] + x[-1]h[1] + x[0]h[0] \quad = (0)1 + (0)3 + (0)1 = 0$$

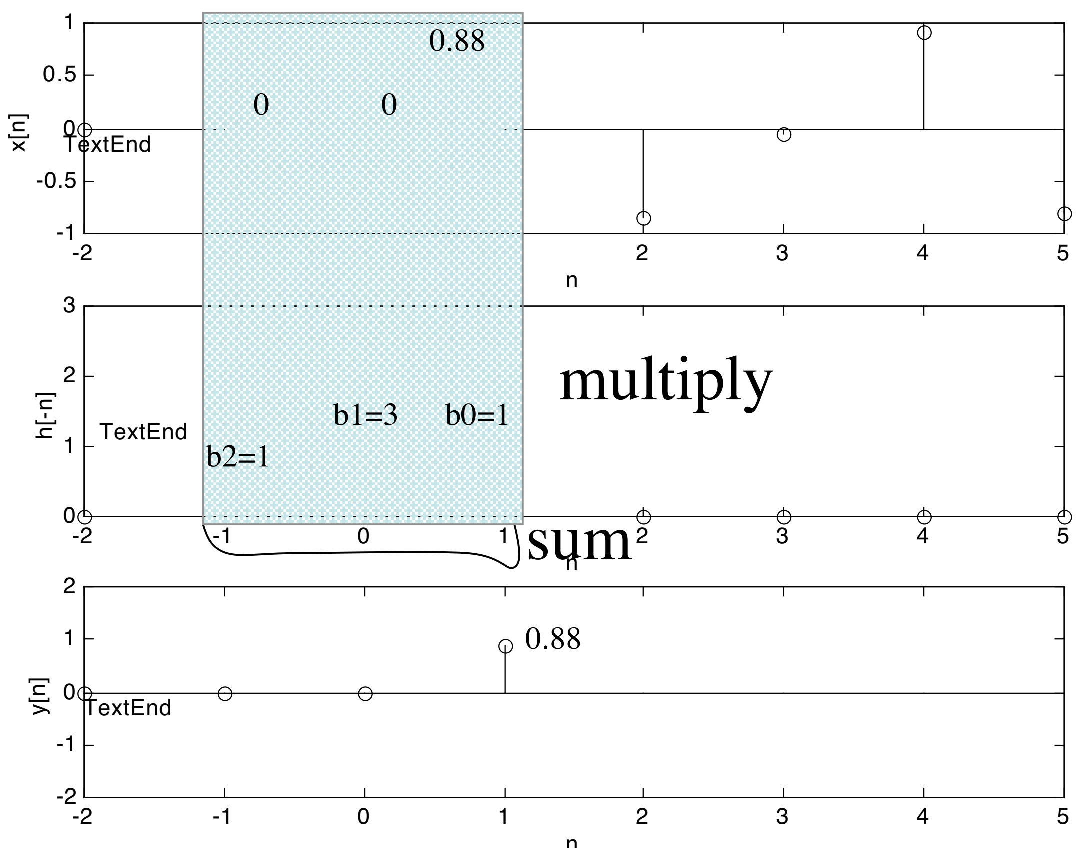
Graphical Convolution

$$y[n] = \sum_{k=-M}^{0} x[k]h[n-k]$$

sum (→ $\sum$), multiply (→ $x[k]h$), shift (→ $[n$), flip (→ $-k$)

n=1

flip/
shift by n



$$y[1] = x[-1]h[2] + x[0]h[1] + x[1]h[0] \;\; = (0)1 + (0)3 + (0.88)1 = 0.88$$

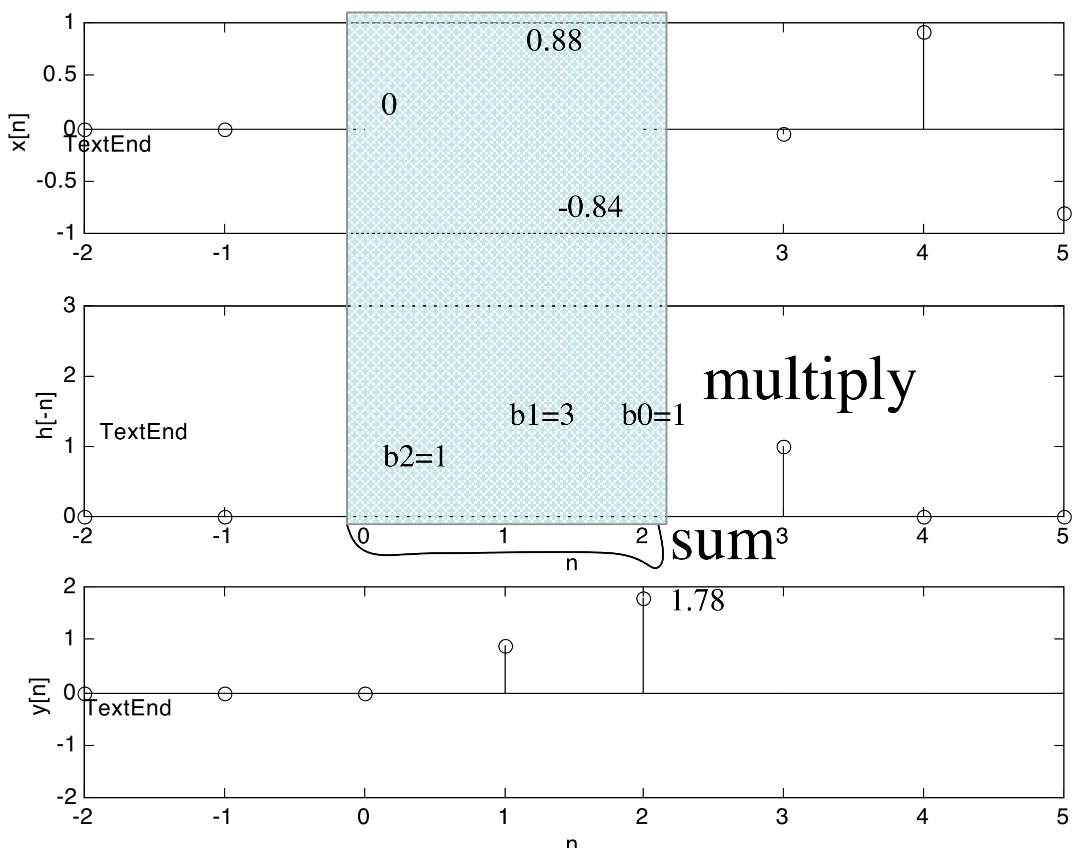
Graphical Convolution

$$y[n] = \sum_{k=-M}^{0} x[k]h[n-k]$$

sum, multiply, shift, flip

n=2

flip/
shift by n



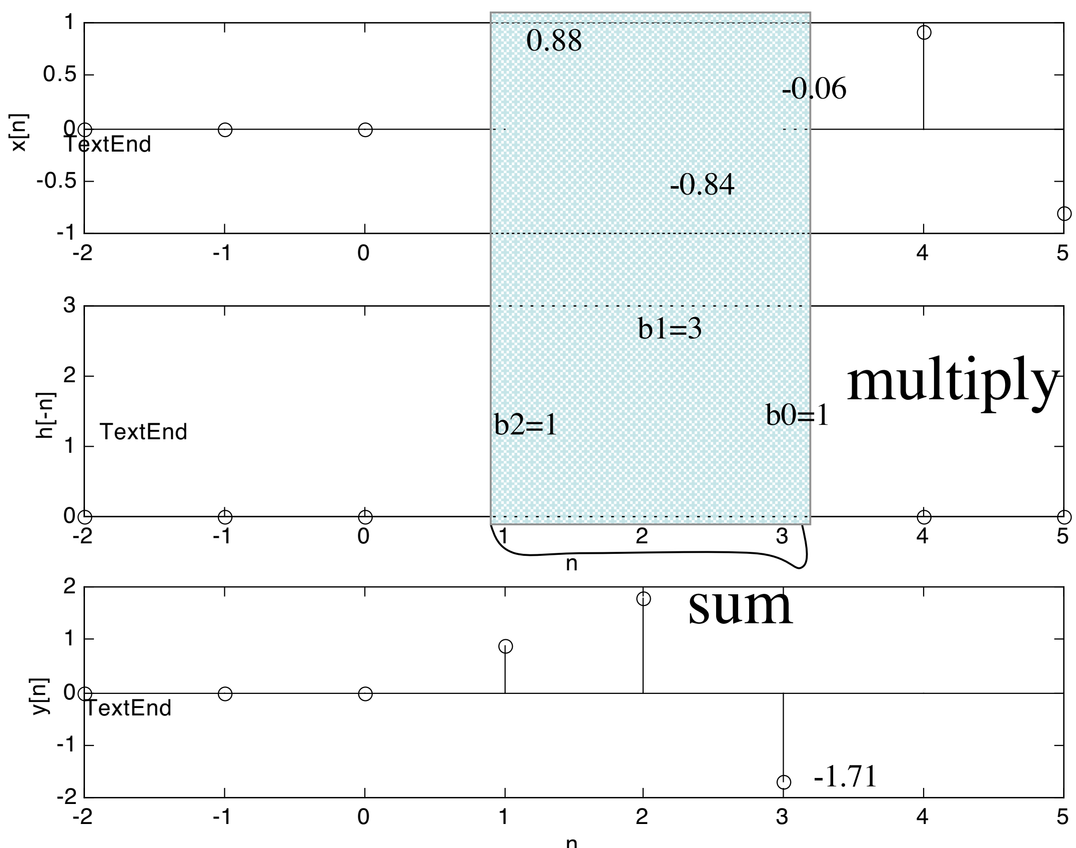
$$y[2] = x[0(\ )]h[2] + x[1]h[1] + x[2]h[0] = (0)1 + (0.88)3 + (-0.84)1 = 1.78$$

Graphical Convolution

$$y[n] = \sum_{k=-M}^{0} x[k]h[n-k]$$

sum, multiply, shift, flip

n=3



flip/
shift by n

multiply

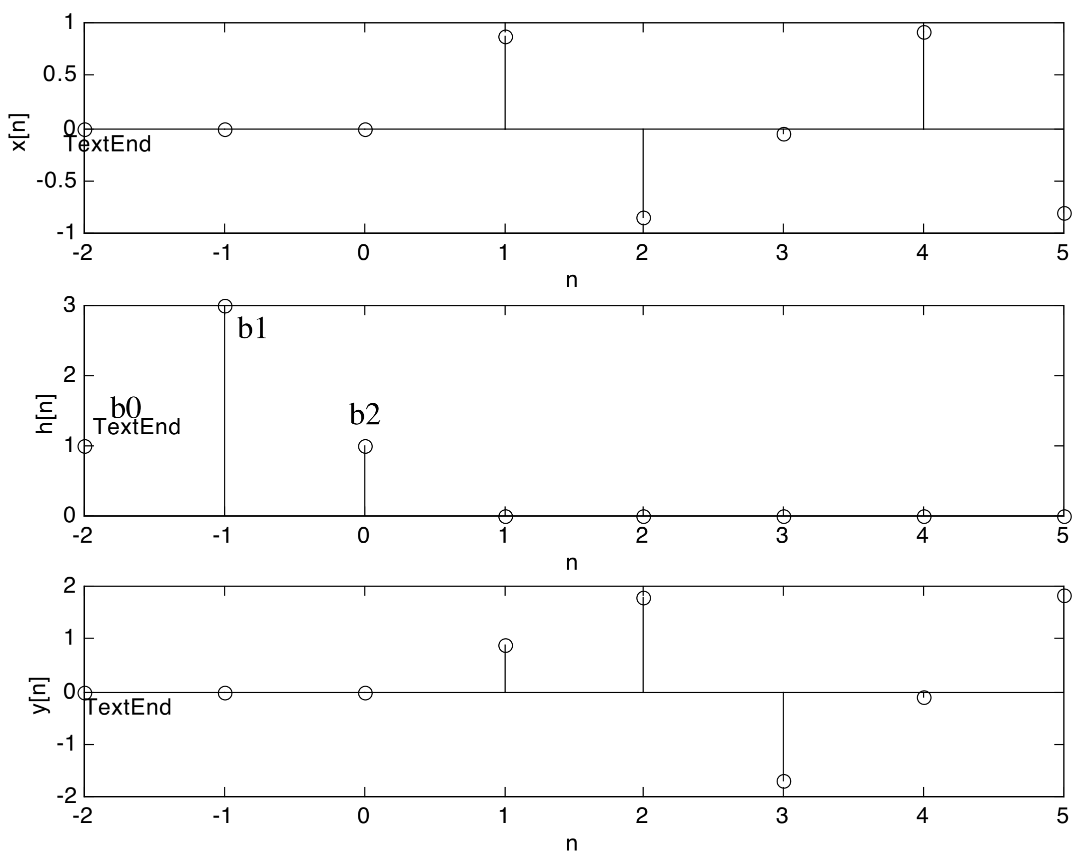
sum

$$y[3] = x[1]h[2] + x[2]h[1] + x[3]h[0] = (0.88)1 + (-0.84)3 + (-0.06)1 = -1.72$$

Graphical Convolution

$$y[n] = \sum_{k=-M}^{0} x[k]h[n-k]$$

$$y[n] = x[k] * h[k]$$



Graphical convolution by decomposition

1. Remember impulse response

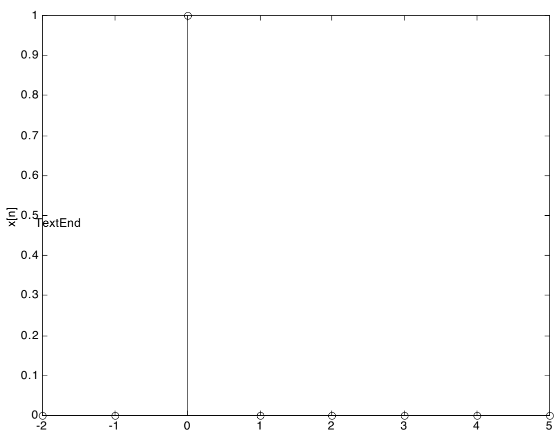
$$x[n] = \delta[n]$$

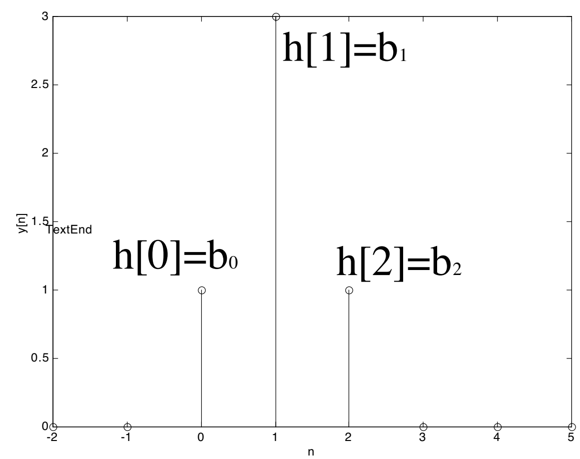
causal FIR filter

b_0, b_1, b_2

$h[n]$

$$h[n] = b_0\delta[n] + b_1\delta[n-1] + b_2\delta[n-2]$$





Graphical convolution by decomposition

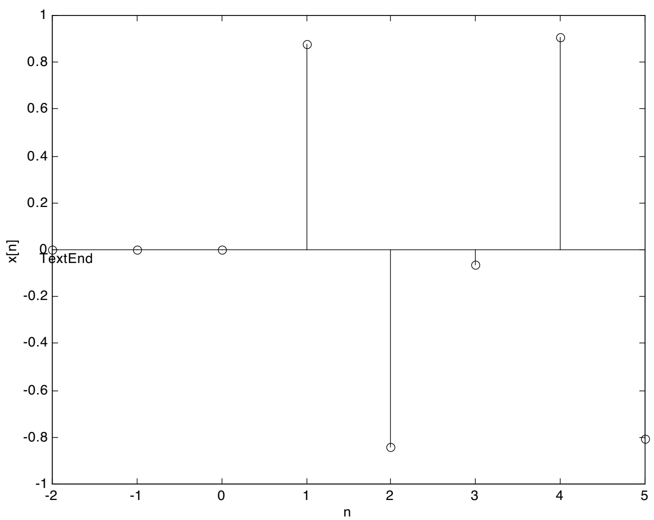
2. Decompose input into sum of scaled delayed impulses

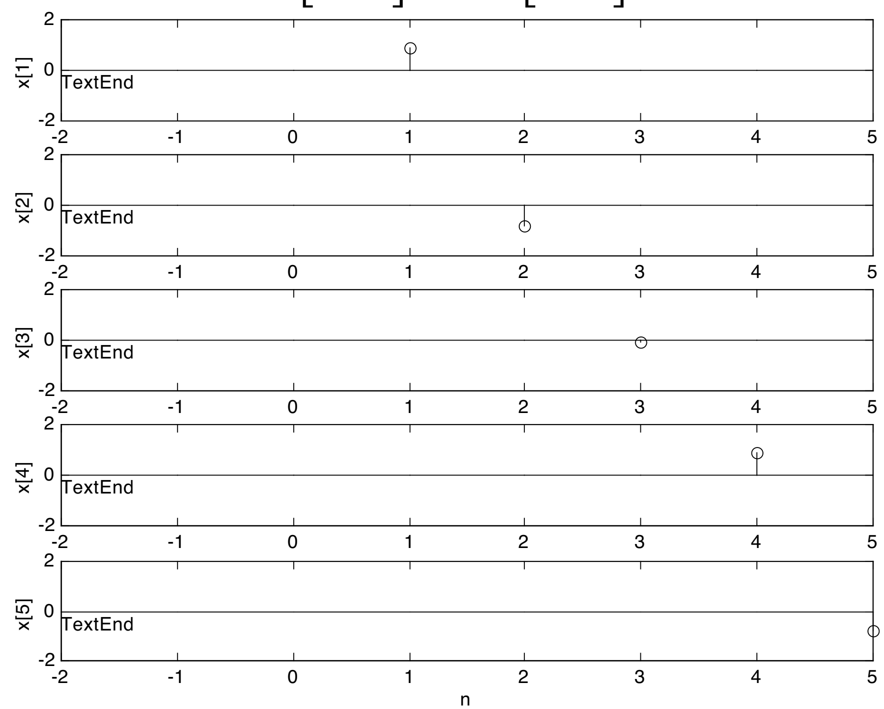
Input

Input as impulses

$$x[n] = 0.88\delta[n-1] - 0.84\delta[n-2] - 0.06\delta[n-3]$$
$$+ 0.90\delta[n-4] - 0.81\delta[n-5]$$

$$x[n] = \sin(2\pi \cdot 0.33n)u[n]$$

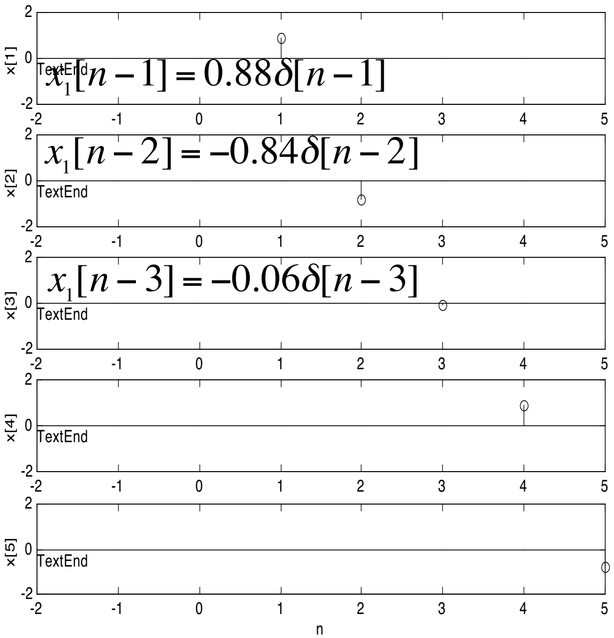




Graphical convolution by decomposition

3. find impulse responses to each impulse

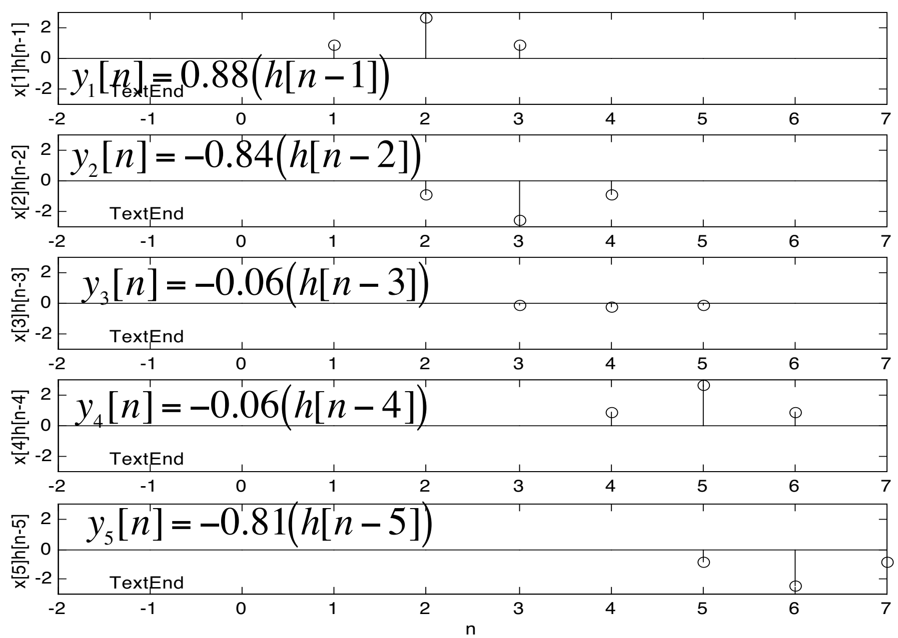
Input as impulses



FIR

Impulse responses

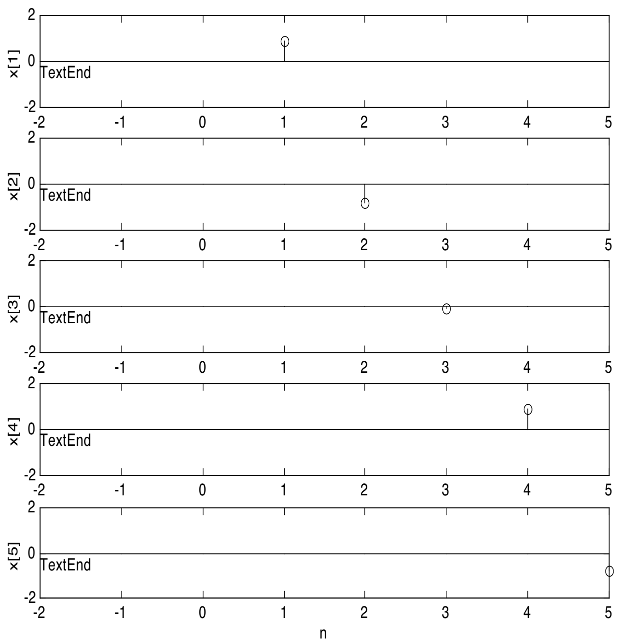
$$h[n]=b_0\delta[n]+b_1\delta[n-1]+b_2\delta[n-2]$$



Graphical convolution by decomposition

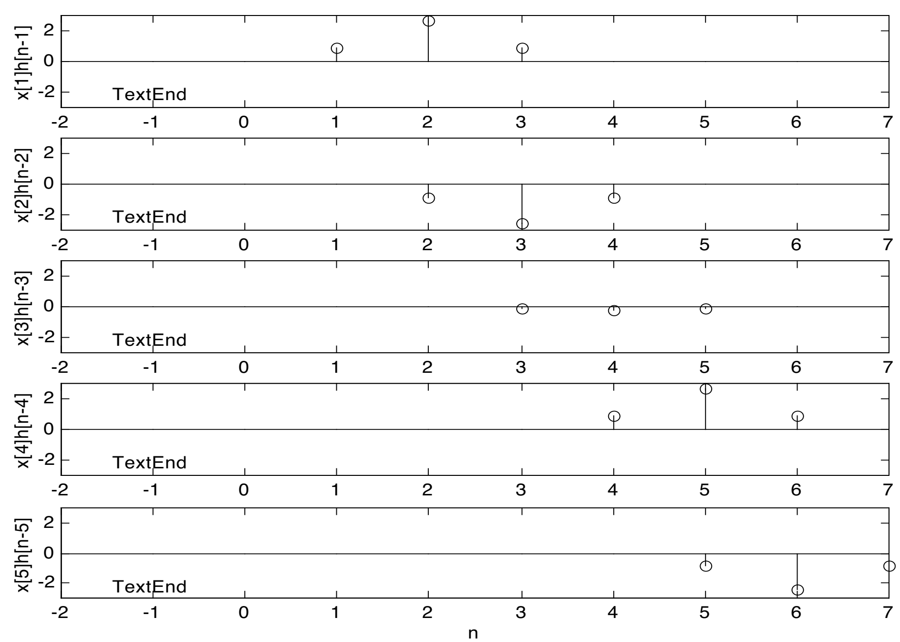
3. sum impulse responses to get total response

Input as impulses



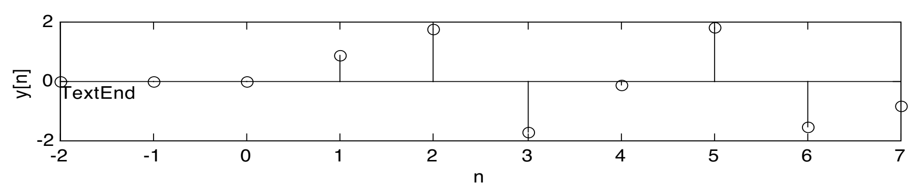
FIR

Impulse responses



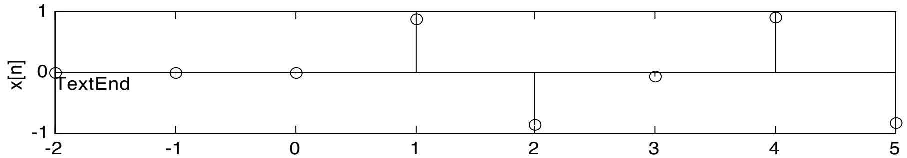
sum

total response

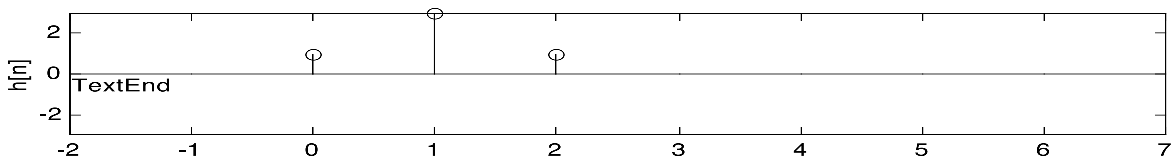


Graphical convolution by decomposition

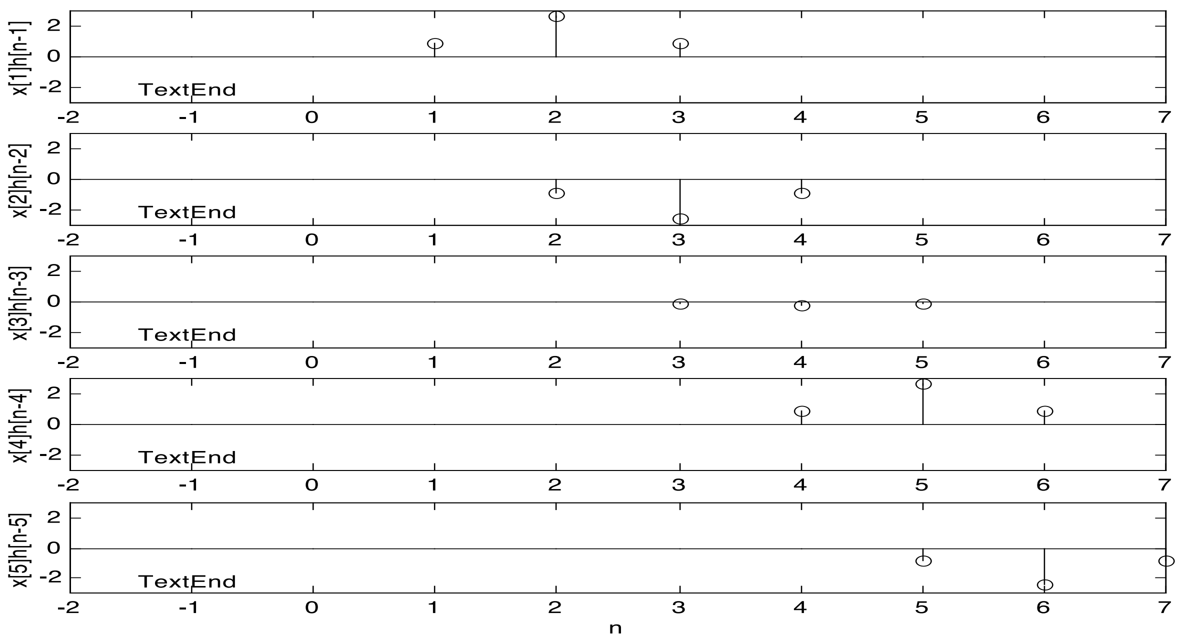
Input



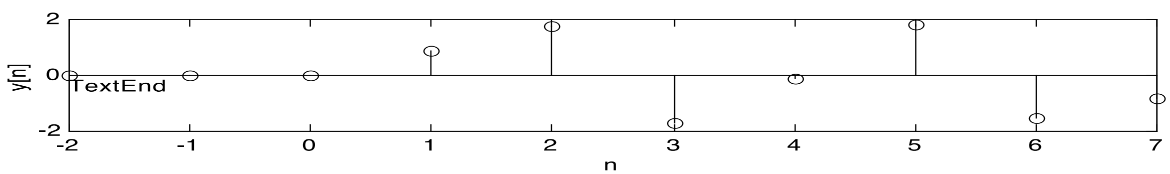
Impulse response



Impulse responses



total response



Synthetic polynomial multiplication

n	-2	-1	0	1	2	3	4	5
x[n]	0	0	0	0.88	-0.84	-0.06	0.90	-0.81
h[n]			1	3	1			
			0	0	0			
				0.88	2.64	0.88		
					-0.84	-2.52	-0.84	
						-0.06	-0.18	-0.06
							0.9	2.7
								-0.81
y[n]	0	0	0	0.88	1.80	-1.7	-0.12	1.83

try $h_1[n] * h_2[n]$ $h_1[n]=[1/3,1/3,1/3]$ $h_2[n]=[1/3,-1/3,1/3]$

Impulse response

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$ FIR filter

$$x[n] = \delta[n] = \begin{cases} 1 & n = 0 \\ 0 & otherwise \end{cases}$$ Delta function

↓

$$y[n]\Big|_{x=\delta[n]} = h[n] = \sum_{k=0}^{M} b_k \delta[n-k]$$

impulse response

$$y[n] = \sum_{k=-\infty}^{\infty} h[n] x[n-k]$$

convolution sum
LTI: FIR, IIR

Frequency response

$$y[n] = \sum_{k=0}^{M} h[k]x[n-k]$$

convolution

$$x[n] = Ae^{j\phi}e^{j\hat{\omega}n}$$

Complex exponential input

$\hat{\omega} = \omega T_s$

$$y[n] = \sum_{k=0}^{M} h[k]Ae^{j\phi}e^{j\hat{\omega}(n-k)}$$

$$= \left(\sum_{k=0}^{M} h[k]e^{-j\hat{\omega}k}\right)Ae^{j\phi}e^{j\hat{\omega}n}$$

$$= H(\hat{\omega})Ae^{j\phi}e^{j\hat{\omega}n}$$

let

$$H(\hat{\omega}) = \sum_{k=0}^{M} h[k]e^{j\hat{\omega}k}$$

$H(\hat{\omega})$ frequency response

$$y[n] = \sum_{k=0}^{M} h[k]x[n-k]$$

convolution

$$x[n] = Ae^{j\phi}e^{j\hat{\omega}n}$$

complex exponential input

$$\downarrow$$

$$y[n] = H(\hat{\omega})Ae^{j\phi}e^{j\hat{\omega}n}$$

$$H(\hat{\omega}) = \sum_{k=0}^{M} h[k]e^{j\hat{\omega}k}$$

frequency response
complex

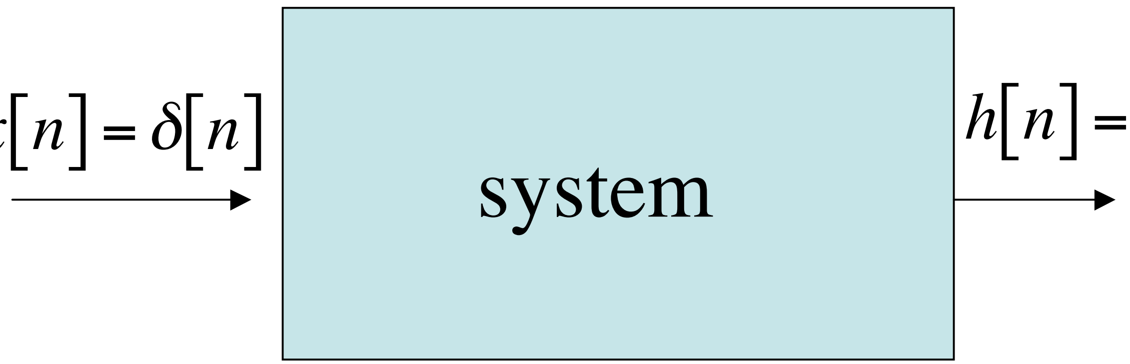
$$y[n] = \left|H(\hat{\omega})\right|Ae^{j(\phi+\angle H(\hat{\omega}))}e^{j\hat{\omega}n}$$

output same frequency
as input, but amplitude scaled
and a phase shift

LTI: FIR & IIR

$$\begin{bmatrix} n & x[n] \\ 0 & 1 \\ 1 & 0 \\ 2 & 0 \\ 3 & 0 \\ 4 & 0 \\ >5 & 0 \end{bmatrix}$$

$$x[n] = \delta[n]$$



$$h[n] = y[n]\big|_{x[n]=\delta[n]}$$

$$\begin{bmatrix} y[n] \\ 1/3 \\ 0 \\ 1/3 \\ 0 \\ 1/3 \\ 0 \end{bmatrix}$$

$$y[n] = ?$$

Ex. $h[n] = \frac{1}{3}\delta[n] + \frac{1}{3}\delta[n-2] + \frac{1}{3}\delta[n-4]$ FIR

$y[n] = \frac{1}{3}x[n] + \frac{1}{3}x[n-2] + \frac{1}{3}x[n-4]$

$$H(\hat{\omega}) = \sum_{k=0}^{4} h[k]e^{-j\hat{\omega}k}$$

$$= h[0]e^{-j\hat{\omega}0} + h[2]e^{-j\hat{\omega}2} + h[4]e^{-j\hat{\omega}4}$$

$$= \tfrac{1}{3} + \tfrac{1}{3}e^{-j\hat{\omega}2} + \tfrac{1}{3}e^{-j\hat{\omega}4}$$

Also try by inspection

$$= \tfrac{1}{3}\left(1 + e^{-j\hat{\omega}2} + e^{-j2\hat{\omega}4}\right) \longleftarrow$$

$$= \tfrac{1}{3}e^{-j\hat{\omega}2}\left(e^{j\hat{\omega}2} + 1 + e^{-j\hat{\omega}2}\right)$$

$$= \tfrac{1}{3}e^{-j\hat{\omega}2}\left(1 + 2\cos 2\hat{\omega}\right)$$

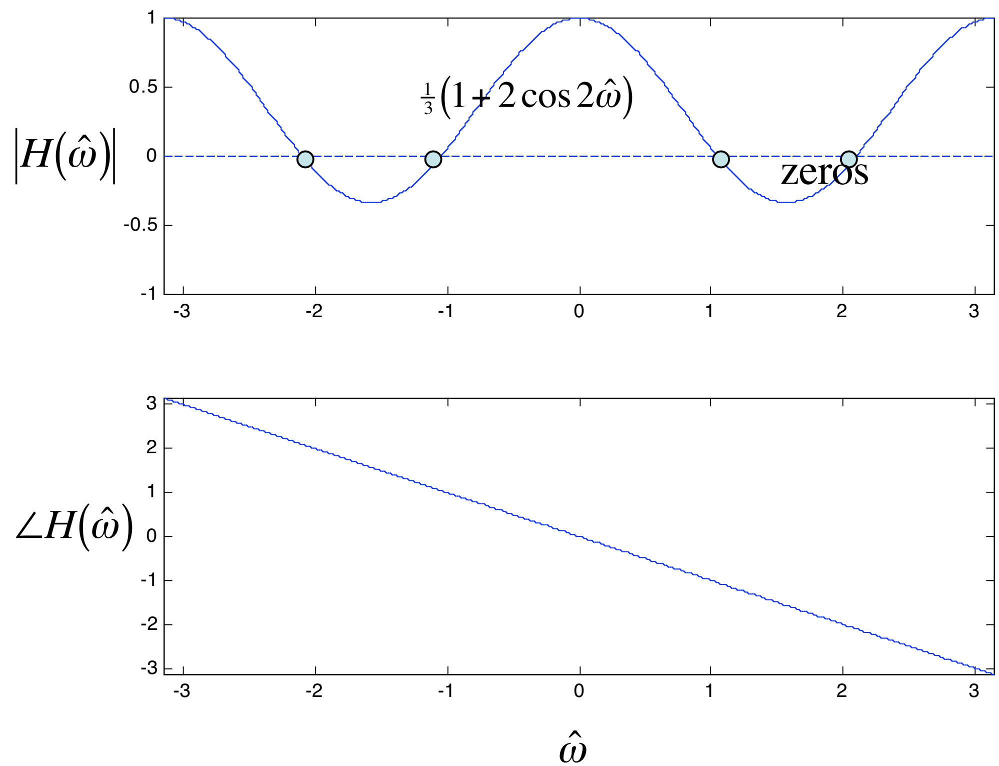
if b_k's symmetric, then factor out $e^{-j\hat{\omega}(M/2)}$ where M is the order of the filter. This leaves complex conjugate paired exponentials to transform into trigonometric functions (cosines/sines).

$$H(\hat{\omega}) = \tfrac{1}{3} e^{-j\hat{\omega}2}(1 + 2\cos 2\hat{\omega})$$

$$|H(\hat{\omega})| = \tfrac{1}{3}|(1 + 2\cos 2\hat{\omega})|$$

Note: $\left|H\left(\tfrac{\pi}{3}\right)\right| = 0$

$\left|H\left(\tfrac{2\pi}{3}\right)\right| = 0$



$\angle H(\hat{\omega}) = -2\hat{\omega}$ linear phase

$\angle H(-\hat{\omega}) = -\angle H(\hat{\omega})$

principal value of phase fn

$-\pi < \angle H(\hat{\omega}) < \pi$ if not, add multiples of 2π

Want positive magnitudes, $|H(\hat{\omega})| \geq 0$
so absorb negative sign into phase by adding
an additional π at each zero

$$H(\hat{\omega}) = \tfrac{1}{3} e^{-j2\hat{\omega}} \left(1 + 2\cos 2\hat{\omega}\right)$$

$$\left|H(\hat{\omega})\right| = \tfrac{1}{3}\left|\left(1 + 2\cos 2\hat{\omega}\right)\right|$$

Note: $\left|H\left(\tfrac{\pi}{3}\right)\right| = 0$

$\left|H\left(\tfrac{2\pi}{3}\right)\right| = 0$

$$\angle H(\hat{\omega}) = -2\hat{\omega}$$

linear phase

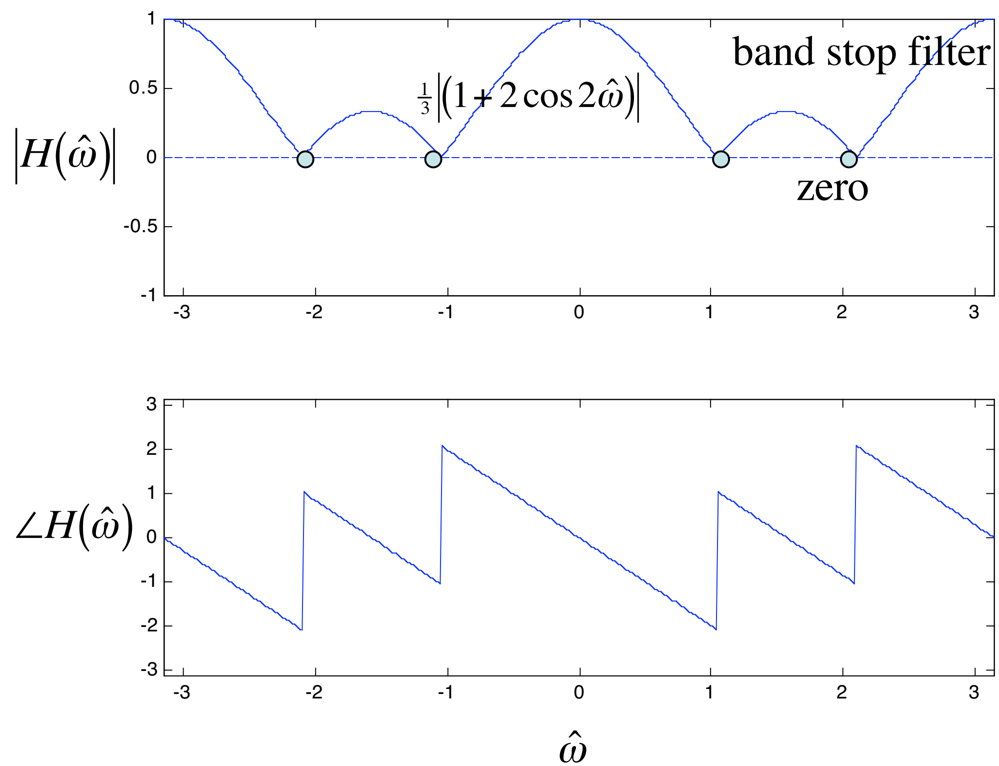
$$= \begin{cases} -2\hat{\omega} & 0 \le \hat{\omega} < \pi/3 \\ -2\hat{\omega} + \pi & \pi/3 \le \hat{\omega} < 2\pi/3 \\ -2\hat{\omega} + 2\pi & 2\pi/3 \le \hat{\omega} < \pi \end{cases}$$

phase odd function

$$\angle H(-\hat{\omega}) = -\angle H(\hat{\omega})$$

principal value of phase fn

$$-\pi < \angle H(\hat{\omega}) < \pi$$



Linear Phase

delay of n_0 sample periods

$$y[n] = x[n - n_0]$$

$$H(\hat{\omega}) = e^{-j\hat{\omega}n_0}$$

$$|H(\hat{\omega})| = 1 \qquad \angle H(\hat{\omega}) = -n_0\hat{\omega}$$ linear phase

FIR filters are linear phase if the coefficients are symmetric

higher frequencies need larger phase shifts than lower frequencies to achieve same time delay

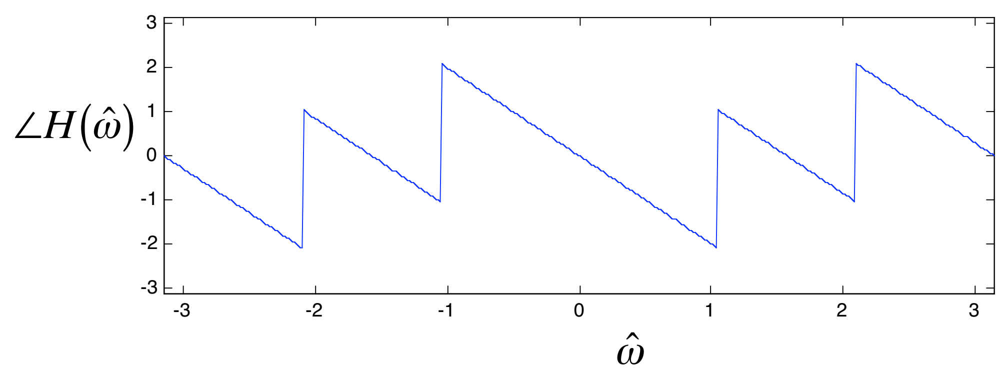
$$\begin{aligned} y &= \sin(2\pi\omega(t + nT_s)) \\ &= \sin(2\pi\omega t + 2\pi\omega n T_s) \\ &= \sin(2\pi\omega t + \phi) \end{aligned}$$

$$\phi = 2\pi T_s n\omega$$

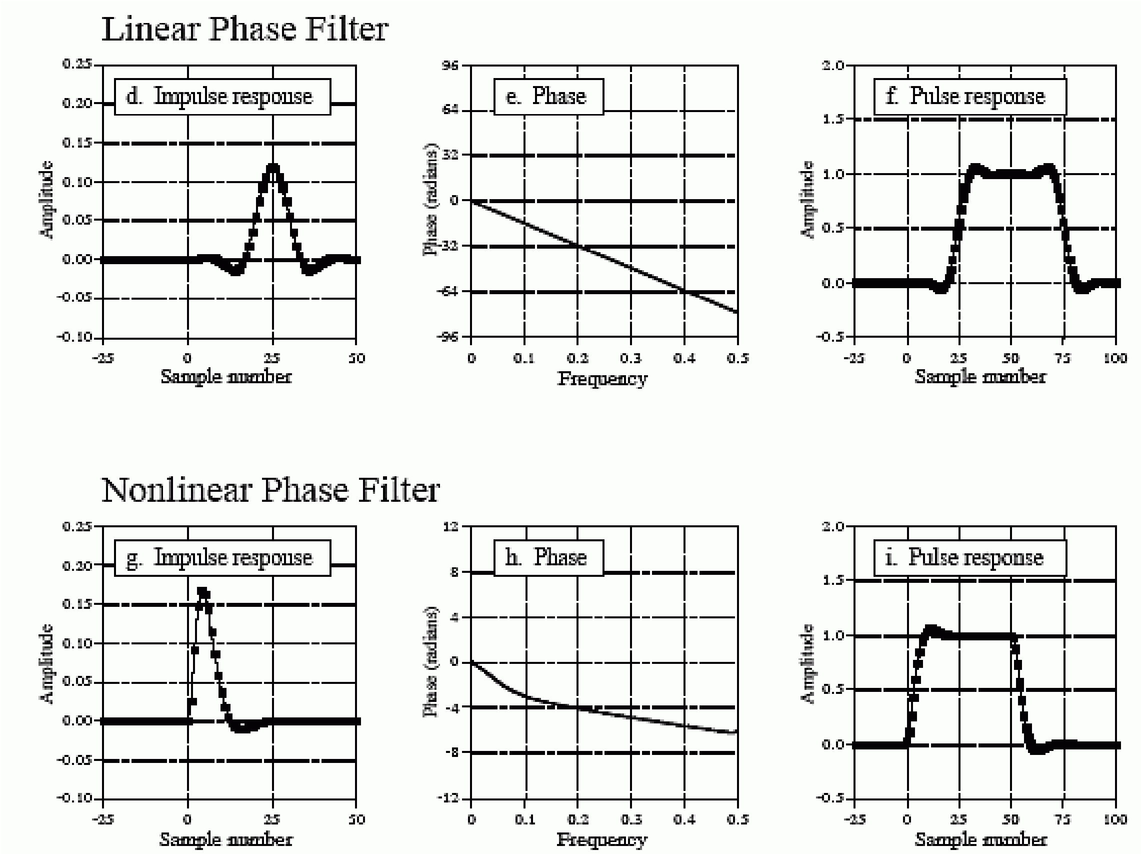
$$\angle H(\hat{\omega}) = -2\hat{\omega}$$

linear phase, 2 sample delay

$$= \begin{cases} -2\hat{\omega} & 0 \le \hat{\omega} < \pi/3 \\ -2\hat{\omega} + \pi & \pi/3 \le \hat{\omega} < 2\pi/3 \\ -2\hat{\omega} + 2\pi & 2\pi/3 \le \hat{\omega} < \pi \end{cases}$$



Linear Phase



"It turns out that, within very generous tolerances, humans are insensitive to [audio] phase shifts. …"– Floyd E. Toole, PhD
Vice President Acoustical Engineering
Harman International Industries, Inc.

"For data transmission, a nonlinear phase delay causes intersymbol interference which increases error rate, particulary if the signal-to-noise ration is poor" – Digital Signal Processing in Communication Systems By Marvin E. Frerking

"These are the pulse responses of each of the filters.The pulse response is nothing more than a positive going step response followed by a negative going step response. The pulse response is used here because it displays what happens to both the rising and falling edges in a signal.

Here is the important part: zero and linear phase filters have left and right edges that look the same, while nonlinear phase filters have left and right edges that look different.

Many applications cannot tolerate the left and right edges looking different. One example is the display of an oscilloscope, where this difference could be misinterpreted as a feature of the signal being measured. Another example is in video processing. Can you imagine turning on your TV to find the left ear of your favorite actor looking different from his right ear?"
http://www.dspguide.com/ch19/4.htm

FREQZ Z-transform digital filter frequency response.
When N is an integer, [H,W] = FREQZ(B,A,N) returns the N-point frequency vector W in radians and the N-point complex frequency response vector H of the filter B/A:

```
                                -1                 -nb
       jw  B(z)   b(1) + b(2)z + .... + b(nb+1)z
    H(e) = ---- = ----------------------------
                                -1                -na
           A(z)   1      + a(2)z + .... + a(na+1)z
```

given numerator and denominator coefficients in vectors B and A.
<snip>
FREQZ(B,A,...) with no output arguments plots the magnitude and unwrapped phase of B/A in the current figure window.

$$H(\hat{\omega}) = \tfrac{1}{3} e^{-2j\hat{\omega}} \left(1 + 2\cos 2\hat{\omega}\right) = \tfrac{1}{3} + \tfrac{1}{3} e^{-j\hat{\omega}2} + \tfrac{1}{3} e^{-j\hat{\omega}4}$$

$$z^{-1} = e^{-j\hat{\omega}}$$

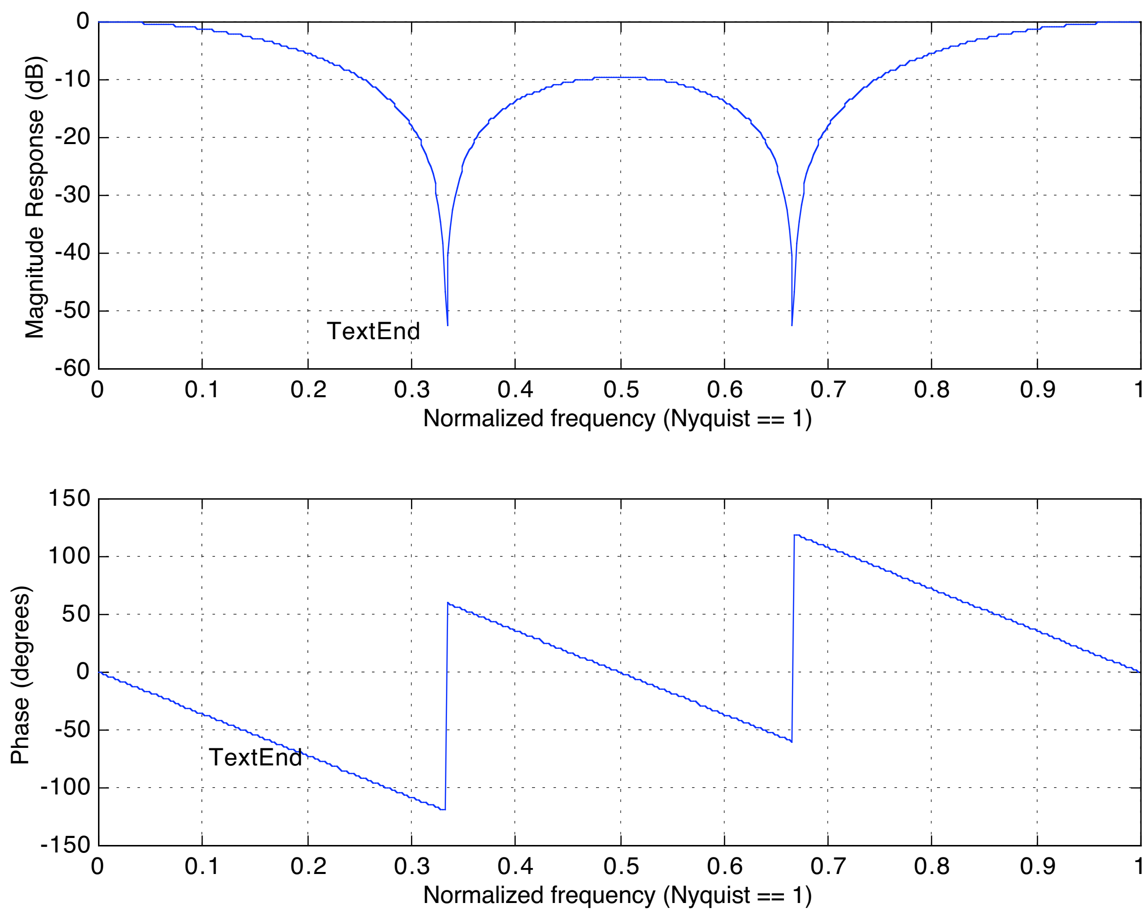
$$H(\hat{\omega}) = \frac{\tfrac{1}{3} + \tfrac{1}{3} z^{-2} + \tfrac{1}{3} z^{-4}}{1} \longrightarrow \begin{array}{l} b(1) = \tfrac{1}{3}, b(2) = 0, b(3) = \tfrac{1}{3}, b(4) = 0, b(5) = \tfrac{1}{3} \\ a(1) = 1 \end{array}$$

```
>> freqz([1/3,0,1/3,0,1/3],[1])
```

$$H(\hat{\omega}) = \tfrac{1}{3}e^{-j2\hat{\omega}}(1+2\cos 2\hat{\omega}) = \tfrac{1}{3} + \tfrac{1}{3}e^{-j\hat{\omega}2} + \tfrac{1}{3}e^{-j\hat{\omega}4}$$

```
>> freqz([1/3,0,1/3,0,1/3],[1])
```

Bode Plot (frequency response curve, amp and phase)



Magnitude response is plotted on a logarithmic scale.

decibels (dB) = $20\log_{10}(|H|)$

Note: In this plot the normalized frequency goes from DC to Nyquist ($\hat{\omega} = \pi$), so this is just one side.
We normally plot from
$-\pi < \hat{\omega} < \pi$.
Remember:
For real filter coefficients, magnitude is an even function; phase is an odd function

Superposition and the frequency response

$$x[n] = 3 + 3\cos(0.6\pi n)$$ input

$$y[n] = \tfrac{1}{3}x[n] + \tfrac{1}{3}x[n-2] + \tfrac{1}{3}x[n-4]$$ FIR filter

sample domain

$$y[n] = 1 + \cos(0.6\pi n) + 1 + \cos(0.6\pi(n-2)) + 1 + \cos(0.6\pi(n-4))$$

$$= 3 + \cos(0.6\pi n)$$

$$+ \cos(0.6\pi n)\cos(1.2\pi) + \sin(0.6\pi n)\sin(1.2\pi)$$

$$+ \cos(0.6\pi n)\cos(2.4\pi) + \sin(0.6\pi n)\sin(2.4\pi)$$

$$= 3 + [1 + \cos(1.2\pi) + \cos(2.4\pi)]\cos(0.6\pi n)$$

$$+ [\sin(1.2\pi) + \sin(2.4\pi)]\sin(0.6\pi n)$$

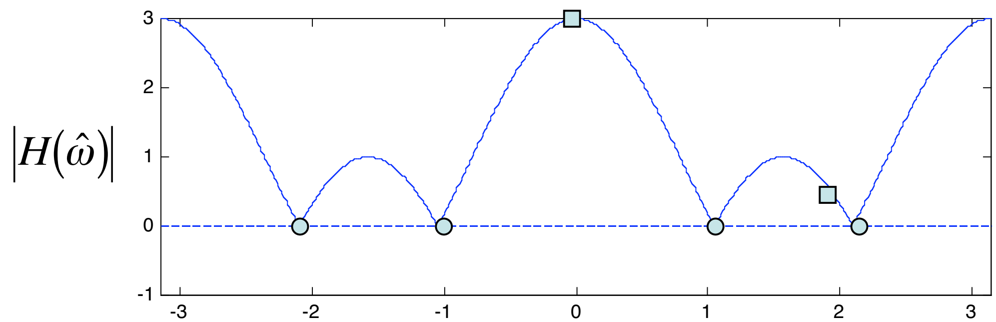
$$= 3 + A\cos(0.6\pi n) + B\sin(0.6\pi n)$$

$$= 3 + \sqrt{A^2 + B^2}\cos(0.6\pi n + \tan^{-1}(B/A))$$

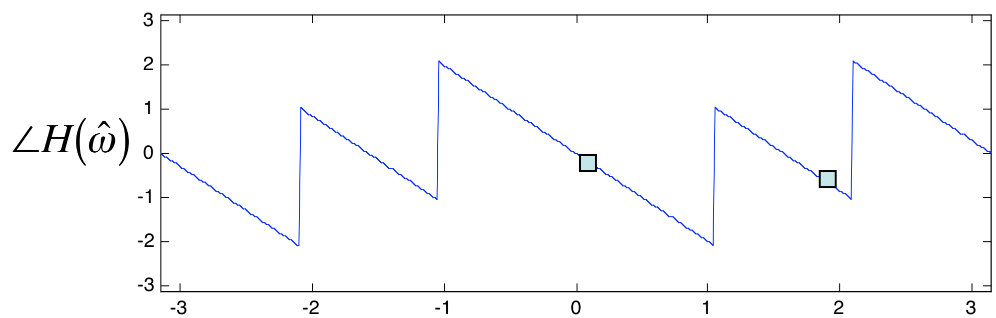
$$= 3 + 0.618\cos(0.6\pi n - 0.2\pi)$$

frequency domain

$$x[n] = 3 + 3\cos(0.6\pi n)$$



closer $\hat{\omega}$ to a zero, the smaller the output.



$\hat{\omega}$

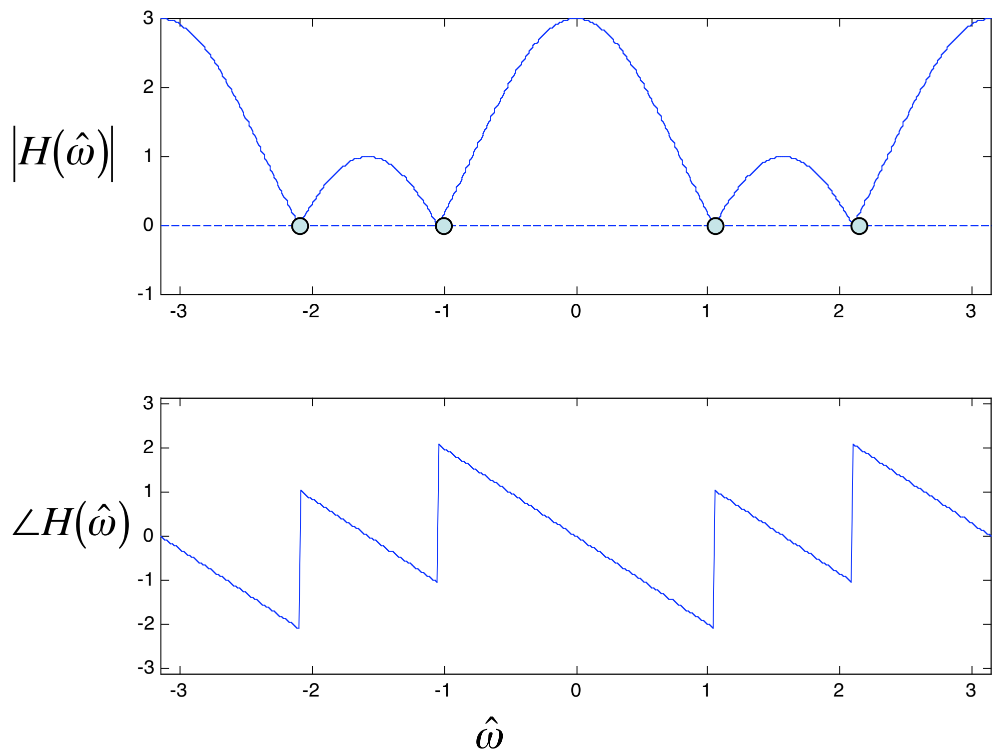
$$\left|H(\hat{\omega})\right| = \tfrac{1}{3}\left(1 + 2\cos 2\hat{\omega}\right) \qquad \left|H(0)\right| = 1 \qquad \left|H(0.6\pi)\right| = 0.206$$

$$\angle H(\hat{\omega}) = -2\hat{\omega} + \pi \qquad \angle H(0) = 0 \quad \angle H(-2 \cdot 0.6\pi + \pi) = -0.2\pi$$

$$y[n] = 3(1) + 3(0.206)\cos(0.6\pi n - 0.2\pi)$$

$$= 3 + 0.618\cos(0.6\pi n - 0.2\pi)$$

chirp



$|H(\hat{\omega})|$
3
2
1
0
-1
-3
-2
-1
0
1
2
3
$\angle H(\hat{\omega})$
3
2
1
0
-1
-2
-3
-3
-2
-1
0
1
2
3
$\hat{\omega}$