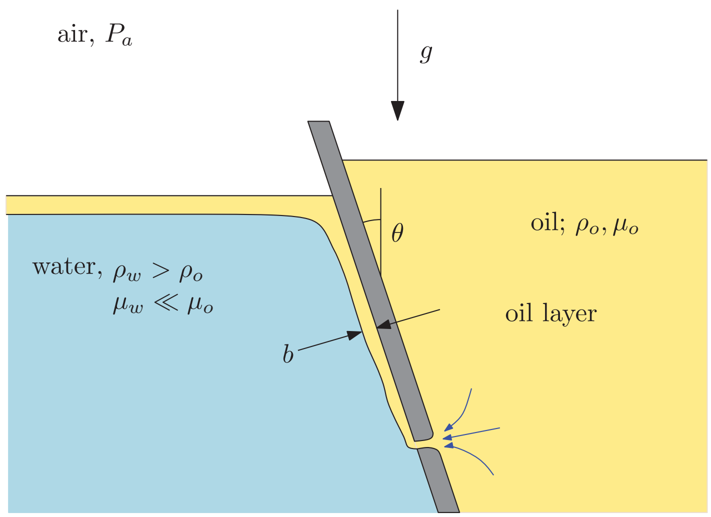
MIT Department of Mechanical Engineering

2.25 Advanced Fluid Mechanics

Problem 6.13

This problem is from "Advanced Fluid Mechanics Problems" by A.H. Shapiro and A.A. Sonin



An oil barge has developed a fine crack in its side, running a length L perpendicular to the sketch. Oil leaks out of the crack and runs up the side of the barge (inclined at an angle θ) in a very thin layer, as sketched. Assume that the flow in the oil layer is highly viscous, that the oil is less dense than the water ($\rho_0 < \rho_w$), and that it is much more viscous than water ($\mu_0 \gg \mu_w$).

(a) If the oil layer is found to have a thickness b, what is the oil volume flow rate Q out through the slit?

(b) Describe qualitatively how the field differs when the viscosity of the water is not negligible compared with the oil viscosity.

Solution:

(a) Since we are told that the oil layer is both thin and viscous, we assume fully developed flow. This, along with assumption that the flow is steady, allows us to ignore all inertial components of the x-momentum equation (Navier-Stokes).

$$0 = -\frac{\partial p}{\partial x} + \mu_o\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right) + \rho_o g_x$$

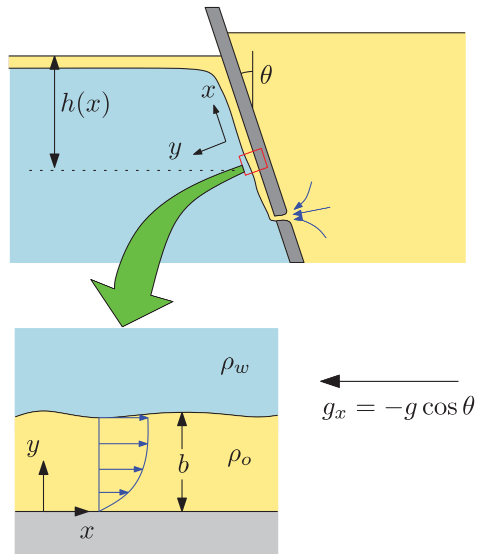
Also because of our thin-film approximation, we can neglect variations in the x-direction ($\frac{d}{dx} \to 0$). Now we have that

$$0 = -\frac{\partial p}{\partial x} + \mu_o\left(\frac{d^2 v_x}{dy^2}\right) + \rho_o g_x$$

Similarly, the y-momentum equation reduces to

$$\frac{\partial p}{\partial y} = 0 \qquad \Rightarrow \qquad p \neq f(y)$$

which tells us that p is only a function of x.



We see from the figure that the pressure at a position x in the oil film is equal to the hydrostatic pressure of the surrounding water. Thus

$$p = \rho_w g h(x) = -\rho_w g(x\cos\theta)$$

$$\Rightarrow \frac{dp}{dx} = -\rho_w g\cos\theta$$

where x has been defined positive up so that a distance h from the free surface is in the $-x$ direction. Now we plug this into the x-momentum equation:

$$0 = \rho_w g\cos\theta + \mu_o\frac{d^2 v_x}{dy^2} - \rho_o g\cos\theta \quad \Rightarrow \quad \frac{d^2 v_x}{dy^2} = -\frac{\rho_w - \rho_o}{\mu_o} g\cos\theta$$

Integrate the expression above twice to find that

$$v_x = -\frac{\rho_w - \rho_o}{2\mu_o} g\cos\theta y^2 + C_1 y + C_2$$

To solve for C_1 and C_2 we must apply boundary conditions. At $y = 0$, we know that $v_x = 0$ such that $C_2 = 0$. At the interface between oil and water ($y = b$) we know that the shear stress must be continuous such that

$$\mu_o \frac{dv_x}{dy}\bigg|_{y=b^-} = \mu_w \frac{dv_x}{dy}\bigg|_{y=b^+}$$

But since we're told that $\mu_w \ll \mu_o$, we can say that $\frac{dv_x}{dy}\big|_{y=b^-} \approx 0$. Therefore,

$$\frac{dv_x}{dx}\bigg|_{y=b} = -\frac{\rho_w - \rho_o}{\mu_o} g\cos\theta b + C_1 = 0 \qquad \Rightarrow \qquad C_1 = \frac{\rho_w - \rho_o}{\mu_o} g\cos\theta b$$

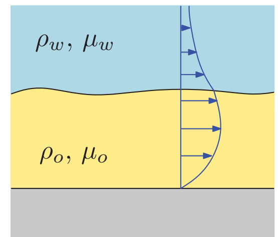
Putting it all together we find

$$v_x = g\cos\theta\left(\frac{\rho_w - \rho_o}{\mu_o}\right)\left(by - \frac{y^2}{2}\right)$$

The volume flow rate is thus

$$\boxed{Q = \int_0^b v_x dy = g\cos\theta \left(\frac{\rho_w - \rho_o}{\mu_o}\right)\frac{b^3}{3}}$$

(b) If the viscosity of the water were not negligible, we would have a shear stress at the interface and the velocity at the interface would be retarded such that the maximum velocity would be within the oil layer (as opposed to being at the interface when $\mu_o \gg \mu_w$).



$\square$

Problem Solution by Tony Yu, Fall 2006

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