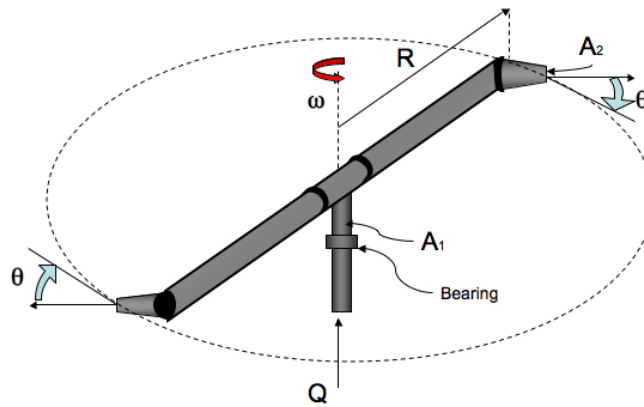


MIT Department of Mechanical Engineering
2.25 Advanced Fluid Mechanics

Problem 5.29

This problem is from “Advanced Fluid Mechanics Problems” by A.H. Shapiro and A.A. Sonin

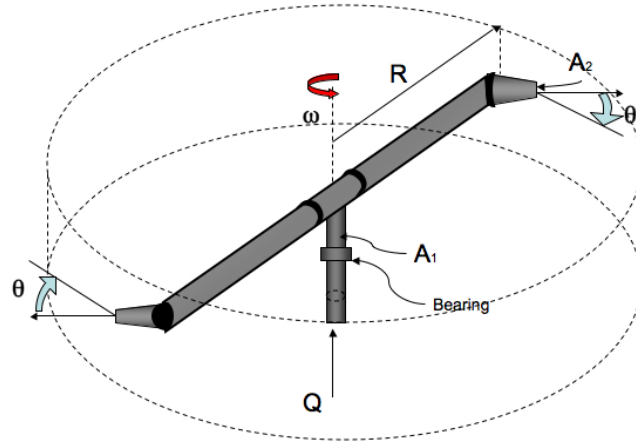


The sketch shows a lawn sprinkler with two horizontal arms of radial length R , at the termination of which are nozzles (exit Area A_2) pointing in a direction which is at an angle θ relative to the tangent of a circumferential line, as shown. The sprinkler is free to rotate, but the bearing on which it is mounted exerts a torque $k\omega$ in the direction opposing the rotation, ω being the angular rate of rotation. A constant volume flow rate Q passes through the sprinkler, the flow being incompressible at density ρ .

- (a) Find an expression for the steady-state angular velocity ω of sprinkler in terms of the given quantities R , A_2 , θ , Q , ρ , and k .
- (b) In the steady state, what is the velocity vector of the fluid emerging from the nozzles, as seen by an observer in the non-rotating reference frame? What is the fluid velocity at the nozzle vent if the bearing is frictionless ($k = 0$)?
- (c) If the pipe area at station 1 near the bearing is A_1 , and the flow from that point to the nozzles is inviscid, what gage pressure is required at station 1 to sustain the flow rate in this steady state?

Solution:

- (a)



Approaching the problem from the laboratory reference frame and taking our CV as fluid plus sprinkler, we have

$$\underbrace{\frac{\partial}{\partial t} \left(\int_{CV} \rho \underline{r} \times \underline{V} dVol \right)}_{\text{steady, } \Rightarrow 0} + \int_{CS} \rho \underline{r} \times \underline{V} (\underline{V} - \underline{V}_{CS}) \cdot \hat{n} dA = k\omega. \quad (5.29a)$$

The angular momentum theorem is applied here in a direction opposite to ω . At the exit of each horizontal arm, we have a tangential velocity $V_\theta = \frac{Q}{2A_2} \cos \theta - \omega R$, then

$$2\rho \left[\left(\frac{Q}{2A_2} \cos \theta - \omega R \right) \times R \right] \frac{Q}{2} = k\omega, \quad (5.29b)$$

then

$$\omega = \frac{Q \cos \theta}{2A_2 \left(R + \frac{k}{\rho Q R} \right)}. \quad (5.29c)$$

Approaching from a rotating reference frame, and taking the same CV, we have

$$\underbrace{\frac{\partial}{\partial t} \left(\int_{VC} \rho \underline{r} \times \underline{V} dV \right)}_{\text{steady, } \Rightarrow 0} + \int_{CS} \rho \underline{r} \times \underline{V} (\underline{V} - \underline{V}_{CS}) \cdot \hat{n} dA = k\omega + T_{Fict}, \quad (5.29d)$$

But now, $V_r = \frac{Q}{2A_2} \cos \theta$ only! Hence,

$$2\rho \left[\left(\frac{Q}{2A_2} \cos \theta \right) \times R \right] \frac{Q}{2} = k\omega + \int \rho 2\omega r v_r dV. \quad (5.29e)$$

Hence, $dV = A_{arm} dr$ and inside the arm $v_r A_{arm} = \frac{Q}{2}$. Hence, evaluating T_{Fict} for both the arms we get the angular momentum equation as

$$\frac{\rho Q^2 R \cos \theta}{2A_2} = k\omega + 2\rho\omega Q \int_0^R r dr = k\omega + \rho\omega QR^2. \quad (5.29f)$$

Rearranging, again we get

$$\omega = \frac{Q \cos \theta}{2A_2(R + \frac{k}{\rho QR})}. \quad (5.29g)$$

- (b) We have the velocity vector emerging from the sprinkler as:

$$\underline{V}_e = \left(\frac{Q}{2A_2} \sin \theta \right) \hat{e}_r + \left(\omega R - \frac{Q}{2A_2} \cos \theta \right) \hat{e}_\theta. \quad (5.29h)$$

If the bearing is frictionless, the angular momentum equation (5.29b) would give $\omega R - \frac{Q}{2A_2} \cos \theta = 0$. Hence the velocity is

$$\underline{V} = \frac{Q}{2A_2} \sin \theta \hat{e}_r. \quad (5.29i)$$

- (c) We have to apply Bernoulli from station (1) to nozzle exit. But for that, we should move to non-inertial rotating reference frame, since only then can we identify a stationary point in space on the nozzle exit! Applying Bernoulli from station 1 to nozzle exit in the rotating frame, we have ¹

$$p_a + \frac{1}{2}\rho V_{e,rel}^2 - \frac{1}{2}\rho\omega^2 R^2 = p_1 + \frac{1}{2}\rho V_1^2, \quad (5.29j)$$

where in the rotating reference frame, we simply have $V_{e,rel}^2 = \left(\frac{Q}{A_2}\right)^2$, and the additional $\frac{1}{2}\rho\omega^2 R^2$ term arises from the pressure rise associated with the centrifugal acceleration of the fluid. (If you find this term puzzling, think about the result in the case when the sprinkler is rotating, but the inlet at station 1 has been blocked so that there is no radial flow in the pipe. It results from $\frac{\partial p}{\partial r} = \rho V_\theta^2/r = \rho r\omega^2$.) Hence, gauge pressure p_{1g} is

$$p_{1g} = \frac{1}{2}\rho \left[\left(\frac{Q}{2A_2} \right)^2 - \omega^2 R^2 - \left(\frac{Q}{A_1} \right)^2 \right]. \quad (5.29k)$$

Note that if the area A_1 is quite less than the area at the T-junction, then it will almost be like a jet problem where there is a significant internal viscous dissipation. In that case, we cannot apply Bernoulli across the T-junction. We are not given this area ratio in this problem and hence nothing can be said here.

In case the two areas are comparable, our result in part (c) is valid.

□

Problem Solution by MK/MC(Updated 2008)/TJO (Updates 2010), Fall 2008

¹Note that we have neglected the radius of the inlet tube, and in the moving reference frame, the water velocity at the inlet is practically the same as in the non rotating reference frame.

MIT OpenCourseWare
<http://ocw.mit.edu>

2.25 Advanced Fluid Mechanics
Fall 2013

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.