13.42 04/01/04: Morrison's Equation

SPRING 2004

1. General form of Morrison's Equation

Flow past a circular cylinder is a canonical problem in ocean engineering. For a purely inviscid, steady flow we know that the force on any body is zero (D'Allembert's paradox). For unsteady inviscid flow this is no longer the case and added mass effects must be considered. Of course in the "real" world, viscosity plays a large role and we must consider, in addition to added mass forces, viscous drag forces resulting from separation and boundary layer friction.

Following on Tuesday's lecture, the resulting force on a body in an unsteady viscous flow can be determined using Morrison's Equation, which is a combination of an inertial term and a drag term. The force in the x-direction on a body in unsteady flow with velocity U(t) is

$$F_x(t) = D(t) = \rho C_m \forall \dot{U} + \frac{1}{2}\rho C_d A U|U| \tag{1.1}$$

In order to obtain rough estimates of the magnitude of the force of a body, it is advantageous to use Morrison's equation with constant coefficients. Supposing we want to find the estimates of the wave forces on a fixed structure, then the procedure would be as follows:

1.) Select an appropriate wave theory (linear waves, or other higher order if necessary).
2.) Select the appropriate C_M and C_D based on Reynolds number, and other factors (see table below).
3.) Apply Morrison's Equation

Wave Theory	C_d	C_m	Comments	Reference
Linear Theory	1.0	0.95	Mean values for ocean wave data on 13-24in cylinders	Wiegel, et al (1957)
	1.0-1.4	2.0	Recommended design values based on statistical analysis of published data	Agerschou and Edens (1965)
Stokes 3rd order	1.34	1.46	Mean Values for oscillatory flow for 2-3in cylinders	Keulegan and Carpenter (1958)
Stokes 5th order	0.8-1.0	2.0	Recommended values based on statistical analysis of published data	Agerschou and Edens (1965)

We can see from the above table that for linear waves the recommended values for drag and mass coefficients are 1.0-1.4 and 2.0, respectively. The range of drag coefficients allows us to account for roughness and Reynolds number effects. These values are for rough estimates. In reality these coefficients vary widely with the various flow parameters and with time. Bretschneider showed that the values of C_D and C_M can even vary over one wave cycle. Even if we ignore the time dependence of these coefficients we must account for the influence of other parameters.

Reynolds number and roughness effects: For smooth cylinders at Reynolds numbers around 10^5, laminar flow transitions to turbulent flow, and there is a dip in C_D as a function of Re. For larger Reynolds numbers the separation point remains essentially constant and thus so does the drag coefficient. In this range C_D is Reynolds number independent. Roughness causes the change from laminar to turbulent flow at a lower Reynolds number and increases the friction and causes a larger C_D. The mass coefficient is influenced by the changes in the boundary layer and is thus also affected Reynolds number and roughness.

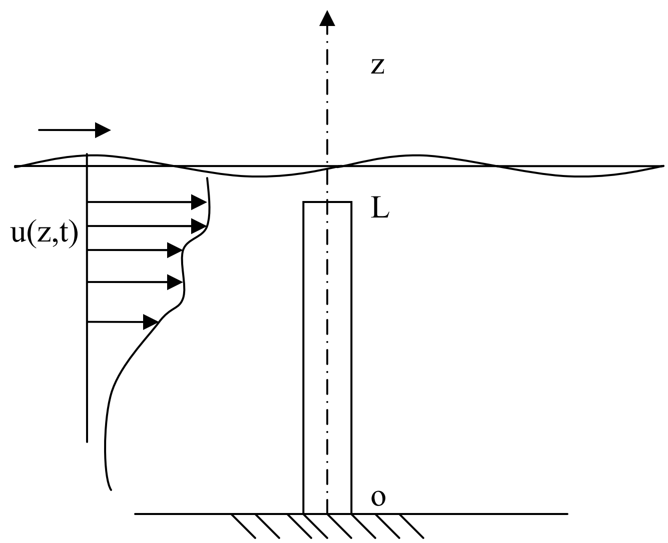

Figure 1. Cylinder in non-uniform inflow

Suppose a vertical cylinder is subject to a current with a horizontal velocity changing both in time and vertically in the z-direction: $u(z,t)$. The approach in practice is to evaluate, using Morrison's formula, the force acting on a small section of the cylinder (eq. 1.2) at each depth and then integrate to get the total force (eq. 1.3).

$$dF(z,t) = C_M \rho \frac{\pi}{4} d^2 \dot{u}(z,t) dz + C_D d \frac{1}{2} \rho u(z,t) |u(z,t)| dz \tag{1.2}$$

$$F(t) = \int_{z=0}^{z=L} dF(z,t) \tag{1.3}$$

The moment on the structure around the origin (point 0) is found by integrating the height z times dF.

$$M(t) = \int_0^L z \, dF(z,t) \tag{1.4}$$

There are several limitations to these integrals. First we are limited to assume that each section does not influence the adjacent sections' flow. This assumption becomes questionable in the case of a cross flow that forces direct interference between the flow of neighboring sections. The second assumption is that the cylinder is not piercing the free surface because in this case the water splashing must be taken into account. In the absence of experimental data we use Morrison's equation as a first estimate in this class, but in reality we should attempt a correction for the surface piercing phenomena.

2. Morrison's Equation when both the body and fluid are moving

Assume that a vertical cylinder is moving with velocity, *u(t)*, within a fluid with velocity, *v(t)*, both velocities in the horizontal direction and uniform in space, then we can write Morrison's equation as follows:

$$F(t) = \rho C_M \frac{\pi}{4} d^2 l\, \dot{v}(t) - \rho\left(C_M - 1\right)\frac{\pi}{4} d^2 h\, \dot{u}(t) + C_D \frac{1}{2}\rho\, d\, l\left[v(t) - u(t)\right]\left|v(t) - u(t)\right| \qquad (2.1)$$

where d is the cylinder diameter and l the cylinder height. It is good to note that this equation does not account for the inertial force due to the mass of the cylinder as required by Newton's law. For example if the cylinder was subject to an inflow we could set $F(t) = m\, a(t)$, where *F(t)* is found using equation 3.1, m is the mass of the cylinder, and *a(t)* is the acceleration.

3. Forces on an Inclined Cylinder

Suppose that a cylinder of diameter, d, and large length, l, is at an angle within an unsteady inflow, *u(t)*, and we would like to use Morrison's equation. It has been suggested that in such cases of slender objects (large l/d) that we can use the following approach: first decompose the inflow velocity into two components U_n and U_t, where U_n is the velocity normal to the cylinder and U_t is the component tangential to the cylinder; then we can use the following expressions:

$$F_n = C_M \rho \frac{\pi}{4} d^2\, \dot{U}_n + C_D \frac{1}{2}\rho\, d\, U_n \left|U_n\right| \qquad (3.1)$$

and

$$F_t = C_f \frac{1}{2}\rho\, \pi d\, U_t \left|U_t\right| \qquad (3.2)$$

to determine the normal and tangential forces per unit length. The mass and drag coefficients are found using the diameter of the cylinder and the normal velocity as in the general (non-inclined) case. The frictional coefficient, C_f, is used in the tangential case instead of the drag coefficient since the drag results from the flow along (tangential) to the cylinder.

In the three-dimensional case where we have a vertical cylinder subject to a velocity vector (*u, v, w*) then we must decompose this velocity into components normal and tangential to the cylinder. The convention is to take the *z*-axis parallel with the cylinder axis (centerline) so that the *w* component is the tangential component of velocity:

$$U_n = \sqrt{u^2 + v^2} \text{ and } U_t = w \tag{3.3}$$

The velocity components (*u, v, w)* used in eq. (3.3) are taken in the coordinate system with z-axis parallel to cylinder axis. Equations (3.1) and (3.2) can then be used to determine the normal and tangential force components. Finally the resultant force in a global coordinate system (say with z-axis perpendicular to the free surface) can be determined using simple trigonometry. These equations have some limitations on as incline angle, as experiments have shown that these expressions are valid up to incline angles of about 60º.

4. Relative Importance of Inertial versus Drag Force Terms

For an incoming wave train with elevation $\eta(x,t) = \frac{h}{2}\cos(kx - \omega t)$ we can determine the relative magnitude of the integrated inertial and drag force terms in Morrison’s equation. Understanding the maximum forces and also when inertial effects dominate over viscous drag, or vice versa, is important in designing offshore structures. The horizontal velocity and acceleration under the wave at the centerline (*x=0*) of the cylinder are

$$u(x=0,z,t) = \frac{h\omega}{2}\frac{\cosh\left[k(z+H)\right]}{\sinh kH}\cos(\omega t) \tag{3.4}$$

$$\frac{\partial u}{\partial t}(0,z,t) = -a\omega^2 \frac{\cosh\left[k\left(z+H\right)\right]}{\sinh kH}\sin\left(\omega t\right) \tag{3.5}$$

Consider the force caused on a vertical cylinder in water depth, *H,* by a linear wave with wavelength, λ, and height, *h*, given in the form

$$F(t) = F_I(t) + F_D(t) \tag{3.6}$$

where $F_I(t)$ is the inertial force term and $F_D(t)$ is the viscous drag force term. The inertial force F_I is dependent on the mass coefficient and can be determined by integrating the force acting over the height of the structure. For a structure fixed to the seafloor the inertial force (in the x-direction) due to waves impinging on a circular cylinder is

$$F_I(t) = -\int_{-H}^{0} \rho C_M \frac{\pi}{4} d^2 \frac{h}{2}\omega^2 \frac{\cosh\left[k\left(z+H\right)\right]}{\sinh\left(kH\right)}\sin\left(\omega t\right)dz\,. \tag{3.7}$$

After integrating eq. (4.4) the resultant inertial force is

$$F_I(t) = -\rho C_M \frac{\pi}{4} d^2 \frac{h}{2k}\omega^2 \sin\left(\omega t\right). \tag{3.8}$$

Similarly we can find the drag force $F_D(t)$ in a similar fashion

$$F_D(t) = \int_{-H}^{0} \frac{1}{2}\rho C_D d\, \frac{h^2}{4}\omega^2 \frac{\cosh^2\left[k\left(z+H\right)\right]}{\sinh^2\left(kH\right)}\cos\left(\omega t\right)\left|\cos\left(\omega t\right)\right| dz \tag{3.9}$$

$$F_D(t) = \frac{1}{2}\rho C_D d\, \frac{h^2\omega^2}{4} \frac{\left\{\frac{\sinh\left(2kH\right)}{4} + \frac{kH}{2}\right\}}{k\sinh^2\left(kH\right)}\cos\left(\omega t\right)\left|\cos\left(\omega t\right)\right| \tag{3.10}$$

The maximum values of these two force components are useful when considering design loads on a structure. One way to determine the maximum is to plot equations (3.8) and (3.10), or simply to look generally at the equations. The maximum inertial force using the result in eq. (3.8) is

$$F_{I_{\max}} = \frac{\pi}{4} C_M \rho d^2 \omega^2 \frac{h}{2k}, \tag{3.11}$$

and the maximum drag force from eq. (4.7) is

$$F_{D_{\max}} = \frac{1}{2}\rho C_d\, d\, \omega^2 \frac{h^2}{4} \frac{\left\{ \frac{\sinh(2kH)}{4} + \frac{kH}{2} \right\}}{k \sinh^2(kH)}. \tag{3.12}$$

To compare the effects of inertial versus drag components we can take the ratio F_I to F_D.

$$\frac{F_{D_{\max}}}{F_{I_{\max}}} = \frac{C_D}{C_M}\frac{h}{d}\left\{ \frac{\sinh(2kH)}{4} + \frac{kH}{2} \right\} \frac{1}{\sinh^2(kH)} \tag{3.13}$$

Substituting the typical values for drag and mass coefficients, 1.0 and 2.0, respectively, allows us to determine how the maximum drag force compares to the maximum inertial force, i.e.

$F_{D_{\max}} > F_{I_{\max}}$ when

$$\frac{h}{d} > 2\pi \frac{4\sinh^2(kH)}{\sinh(2kH) + 2kH} = 2\pi \frac{4\sinh^2\left(2\pi \frac{H}{\lambda}\right)}{\sinh\left(4\pi \frac{H}{\lambda}\right) + 4\pi \frac{H}{\lambda}} \tag{3.14}$$

If $\frac{H}{\lambda}$ is large (deep water) then $\sinh(kH) \simeq \frac{e^{kH}}{2}$ and the deep water asymptote for h/d, such that $F_{D_{\max}} > F_{I_{\max}}$, is

$$\frac{h}{d} \simeq 4\pi \frac{e^{\left(4\pi \frac{H}{\lambda}\right)}}{e^{\left(4\pi \frac{H}{\lambda}\right)} + 4\pi \frac{H}{\lambda}} \approx 4\pi \tag{3.15}$$

If $\frac{H}{\lambda}$ is small (very shallow water) then $\sinh(kH) \simeq kH$ and the shallow water asymptote for h/d, such that $F_{D_{\max}} > F_{I_{\max}}$, is

$$\frac{h}{d} \simeq 2\pi \frac{4\left(2\pi \frac{H}{\lambda}\right)^2}{4\pi \frac{H}{\lambda} + 4\pi \frac{H}{\lambda}} = (2\pi)^2 \frac{H}{\lambda} \tag{3.16}$$

We could also consider the maximum drag and inertial force (force/length) terms at a specific depth, *z*, below the free surface:

$$F_{D_{\max}}(z) = \frac{1}{2}\rho C_D d \frac{h^2}{4}\omega^2 \frac{\cosh^2\left[k(z+H)\right]}{\sinh^2(kH)} \tag{3.17}$$

$$F_{I_{\max}}(z) = \rho C_M \frac{\pi}{4} d^2 \frac{h}{2}\omega^2 \frac{\cosh\left[k(z+H)\right]}{\sinh(kH)} \tag{3.18}$$

The ratio of these two components is

$$\frac{F_{D_{\max}}(z)}{F_{I_{\max}}(z)} = \frac{C_D}{\pi C_M}\frac{h}{d}\frac{\cosh\left[k(z+H)\right]}{\sinh(kH)}. \tag{3.19}$$

It is evident from equation (3.17) that the drag force is most significant nearest to the surface and it decreases with increasing depth. Take as an example, the case of deep water waves with typical values for C_d and C_m (1.0 and 2.0 respectively), the ratio of drag and inertial forces as a function of depth are

$$\frac{F_{D_{\max}}(z)}{F_{I_{\max}}(z)} = \frac{1}{2\pi}\frac{h}{d}e^{kz} \tag{3.20}$$

The maximum total force integrated over the height of the cylinder can be determined using equations (3.8) and (3.10). Thus total force as a function of time is

$$F(t) = -\rho C_M \frac{\pi}{4} d^2 \frac{h}{2k}\omega^2 \sin(\omega t) + \frac{1}{2}\rho C_D d \frac{h^2\omega^2}{4} \frac{\left\{\frac{\sinh(2kH)}{4} + \frac{kH}{2}\right\}}{k \sinh^2(kH)} \cos(\omega t)\left|\cos(\omega t)\right| \tag{3.21}$$

$$F(t) = -F_{Io}\sin(\omega t) + F_{Do}\cos(\omega t)\left|\cos(\omega t)\right| \tag{3.22}$$

Further discussion of Morrison's equation can be found in Environmental Descriptions pgs. 6.27a-l.