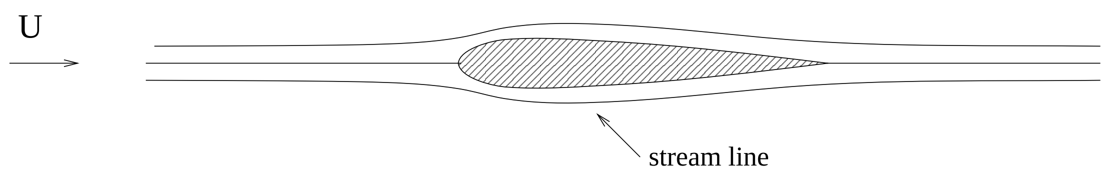
2.20 - Marine Hydrodynamics
Lecture 12

3.14 Lifting Surfaces

3.14.1 2D Symmetric Streamlined Body

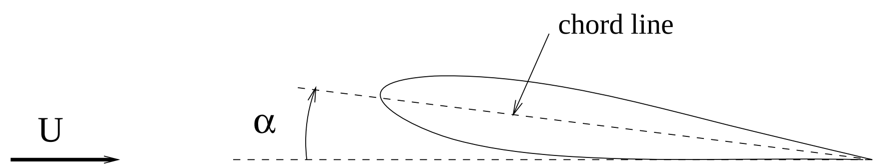
No separation, even for large Reynolds numbers.



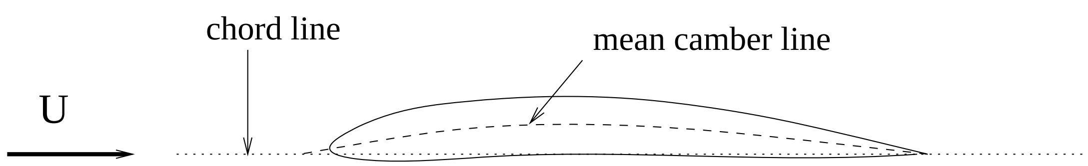
- Viscous effects only in a thin boundary layer.
- Small Drag (only skin friction).
- No Lift.

3.14.2 **Asymmetric Body**

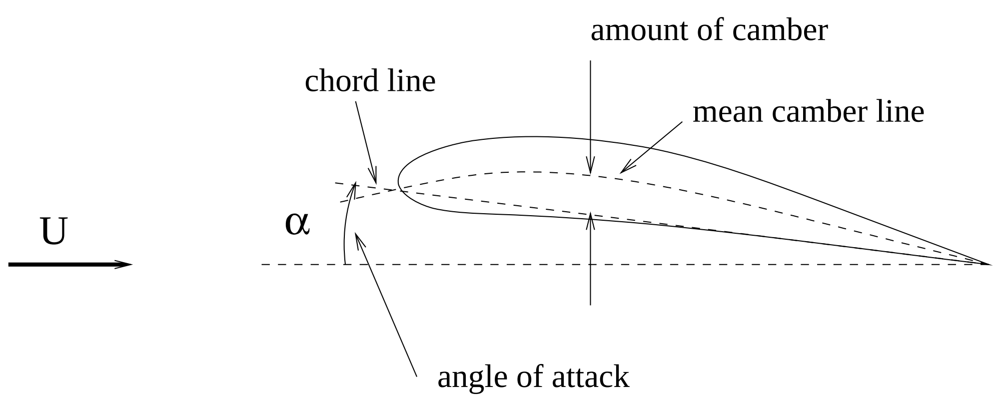
(a) Angle of attack α,



(b) or camber $\eta(x)$,



(c) or both



$$\boxed{\text{Lift} \perp \text{ to } \vec{U} \text{ and Drag} \parallel \text{ to } \vec{U}}$$

3.15 Potential Flow and Kutta Condition

From the P-Flow solution for flow past a body we obtain

P-Flow solution, infinite velocity at trailing edge.

Note that (a) the solution is not unique - we can always superimpose a circulatory flow without violating the boundary conditions, and (b) the velocity at the trailing edge $\rightarrow \infty$. We must therefore, impose the Kutta condition, which states that the **'flow leaves tangentially the trailing edge, i.e., the velocity at the trailing edge is finite'**. To satisfy the Kutta condition we need to add circulation.

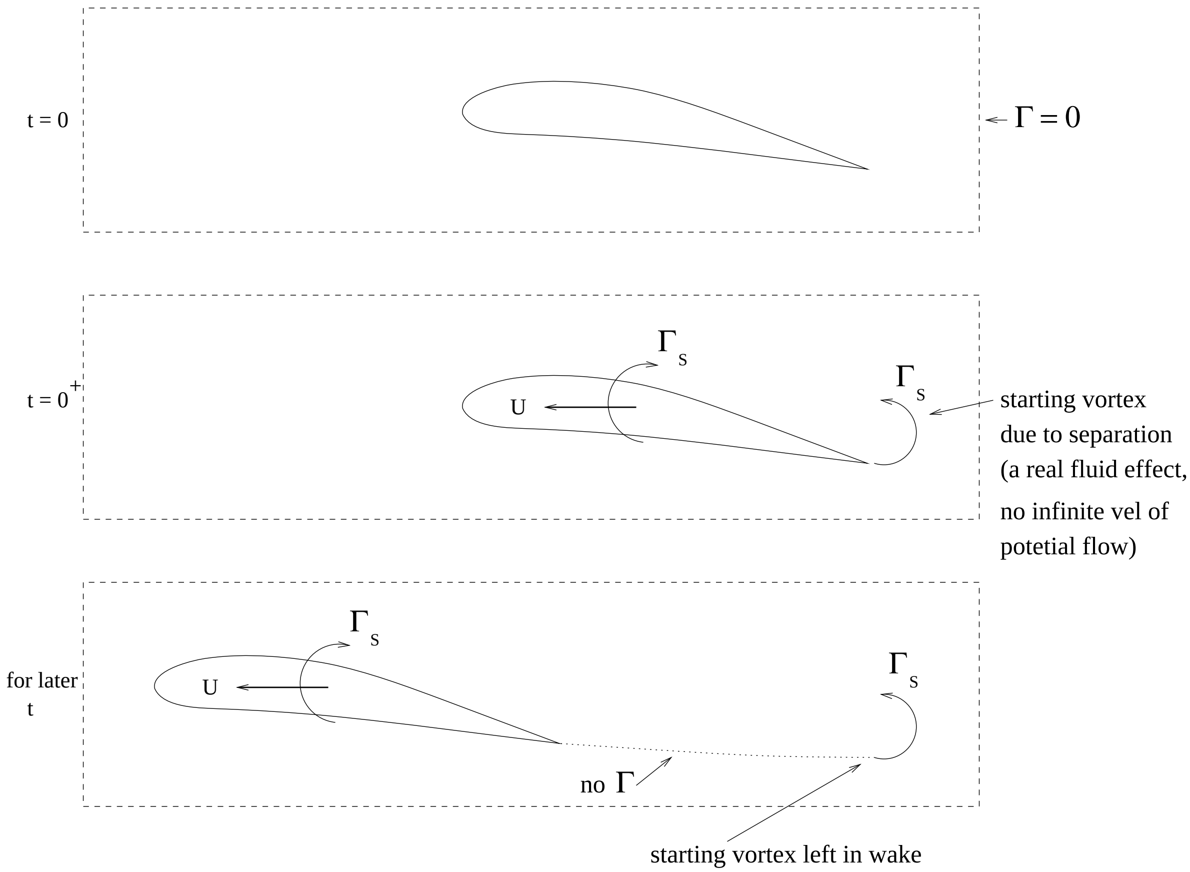
Circulatory flow only.

Superimposing the P-Flow solution plus circulatory flow, we obtain

Figure 1: P-Flow solution plus circulatory flow.

3.15.1 **Why Kutta condition**?

Consider a control volume as illustrated below. At $t = 0$, the foil is at rest (top control volume). It starts moving impulsively with speed U (middle control volume). At $t = 0^+$, a starting vortex is created due to flow separation at the trailing edge. As the foil moves, viscous effects streamline the flow at the trailing edge (no separation for later t), and the starting vortex is left in the wake (bottom control volume).

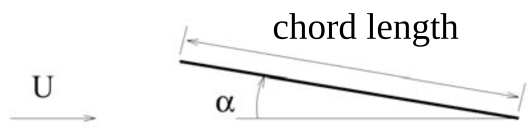


Kelvin's theorem:

$$\frac{d\Gamma}{dt} = 0 \rightarrow \Gamma = 0 \text{ for } t \geq 0 \text{ if } \Gamma(t = 0) = 0$$

After a while the Γ_S in the wake is far behind and we recover Figure 1.

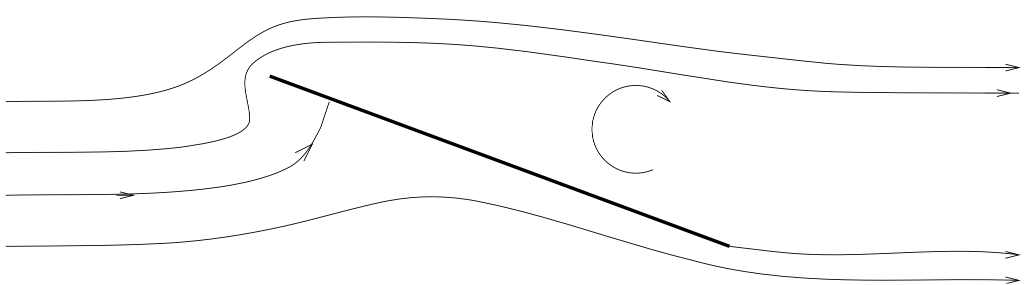
3.15.2 **How much Γ_S?**
Just enough so that the Kutta condition is satisfied, so that no separation occurs. For example, consider a flat plate of chord ℓ and angle of attack α, as shown in the figure below.



Simple P-Flow solution

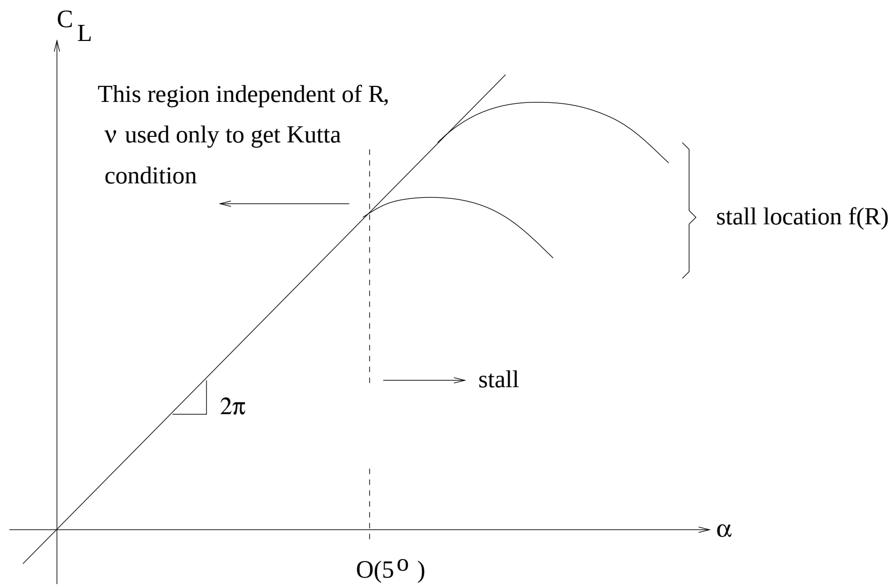
$$\Gamma = \pi l U \sin\alpha$$
$$L = \rho U \Gamma = \rho U^2 \pi l \sin\alpha$$
$$C_L = \frac{|\vec{L}|}{\frac{1}{2}\rho U^2 l} = \underbrace{2\pi\sin\alpha}_{\substack{\text{only for}\\ \text{small }\alpha}} \approx 2\pi\alpha \text{ for small } \alpha$$

However, notice that as α increases, separation occurs close to the leading edge.

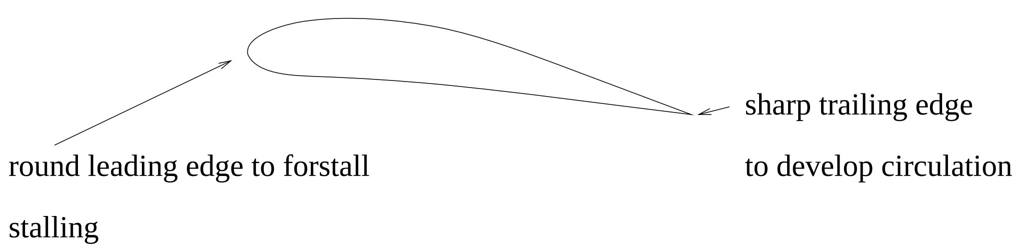


Excessive angle of attack leads to separation at the leading edge.

When the angle of attack exceeds a certain value (depends on the wing geometry) stall occurs. The effects of stalling on the lift coefficient ($C_L = \frac{L}{\frac{1}{2}\rho U^2 \text{ span}}$) are shown in the following figure.



- In experiments, $C_L < 2\pi\alpha$ for 3D foil - finite aspect ratio (finite span).
- With sharp leading edge, separation/stall to early.

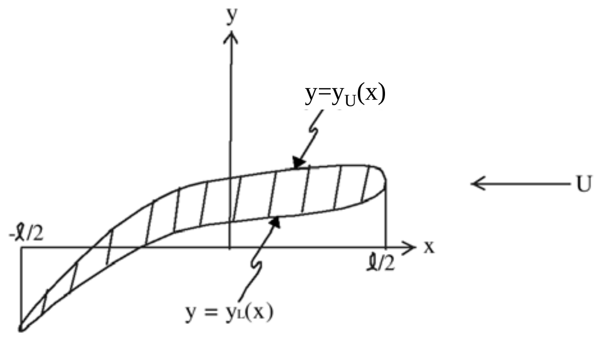


3.16 Thin Wing, Small Angle of Attack

- **Assumptions**

  – **Flow:** Steady, P-Flow.

  – **Wing:** Let $y_U(x)$, $y_L(x)$ denote the upper and lower vertical camber coordinates, respectively. Also, let $x = \ell/2$, $x = -\ell/2$ denote the horizontal coordinates of the leading and trailing edge, respectively, as shown in the figure below.



For thin wing, at a small angle of attack it is

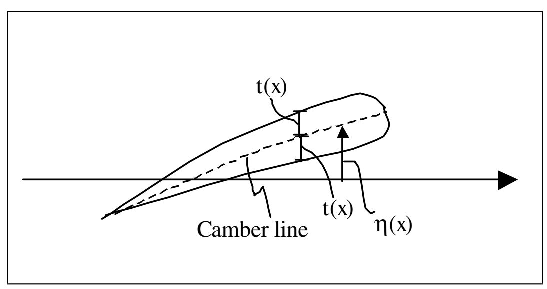
$$\frac{y_U}{\ell},\ \frac{y_L}{\ell} \quad << \quad 1$$
$$\frac{dy_U}{dx},\ \frac{dy_L}{dx} \quad << \quad 1$$

The problem is then linear and superposition applies.
Let $\eta(x)$ denote the camber line

$$\eta(x) = \frac{1}{2}(y_U(x) + y_L(x)),$$

and $t(x)$ denote the half-thickness

$$t(x) = \frac{1}{2}(y_U(x) - y_L(x)).$$

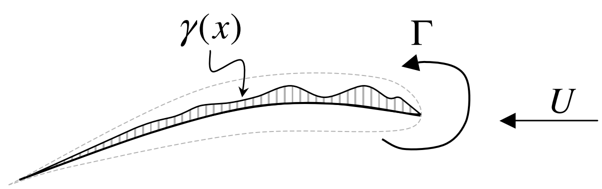


For linearized theory, i.e. thin wing at small AoA, the lift on the wing depends only on the camber line **but not** on the wing thickness. Therefore, for the following analysis we approximate the wing by the camber line only and ignore the wing thickness.

- **Definitions**

In general, the lift on the wing is due to the total circulation Γ around the wing. This total circulation can be given in terms due to a distribution of circulation $\gamma(x)$ (Units: $[LT^{-1}]$) inside the wing, i.e.,

$$\Gamma = \int_{-\ell/2}^{\ell/2} \gamma(x) dx$$



Noting that superposition applies, let the total potential Φ for this flow be expressed as the sum of two potentials

$$\Phi = \underbrace{-Ux}_{\substack{\text{Free stream} \\ \text{potential}}} + \underbrace{\phi}_{\substack{\text{Disturbunce} \\ \text{potential}}}$$

The flow velocity can by expressed as

$$\vec{v} = \nabla\Phi = (-U + u, v)$$

where (u, v) are given by $\nabla\phi = (u, v)$ and denote the velocity disturbance, due to the presence of the wing. For linearized wing we can *assume*

$$u, v << U \;\Rightarrow\; \frac{u}{U}, \frac{v}{U} << 1$$

Consider a flow property q, such as velocity, pressure etc. Then let $q_U = q(x, 0_+)$ and $q_L = q(x, 0_-)$ denote the values of q at the upper and lower wing surfaces, respectively.

- **Lift due to circulation**

Applying Bernoulli equation for steady, inviscid, rotational flow, along a streamline from ∞ to a point on the wing, we obtain

$$
\begin{aligned}
p - p_\infty &= -\frac{1}{2}\rho \left(|\vec{v}|^2 - U^2\right) \Rightarrow \\
p - p_\infty &= -\frac{1}{2}\rho \left\{\left((u-U)^2 + v^2\right) - U^2\right\} = -\frac{1}{2}\rho(u^2 + v^2 - 2uU) \Rightarrow \\
p - p_\infty &= -\frac{1}{2}\rho u U(\underbrace{\frac{u}{U}}_{<<1} + \underbrace{\frac{v}{U}}_{<<1}\underbrace{\frac{v}{u}}_{\sim 1} - 2)
\end{aligned}
$$

Dropping terms of order $\frac{u}{U}$, $\frac{v}{U} << 1$ we obtained the **linearized** Bernoulli equation for thin wing at small AoA

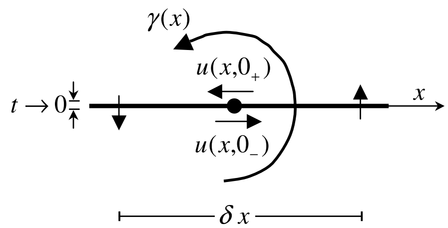
$$p - p_\infty = \rho u U$$

Integrating the pressure along the wing surface, we obtain an expression for the **total lift** L on the wing

$$
\begin{aligned}
L &= \oint (p - p_\infty) n_y dS = \int_{-l/2}^{l/2} \left[\left(p(x, 0_-) - p_\infty\right) - \left(p(x, 0_+) - p_\infty\right)\right] dx \\
L &= \int_{-l/2}^{l/2} \left(p(x, 0_-) - p(x, 0_+)\right) dx = \rho U \int_{-l/2}^{l/2} \left(u(x, 0_-) - u(x, 0_+)\right) dx \qquad (1)
\end{aligned}
$$

To obtain the total lift on the wing we will seek an expression for $u(x, 0_\pm)$.

Consider a closed contour on the wing, of negligible thickness, as shown in the figure below.



In this case we have

$$\gamma(x)\delta x = |u(x,0_+)|\delta x + u(x,0_-)\delta x \Rightarrow \gamma(x) = |u(x,0_+)| + u(x,0_-)$$

For small u/U we can argue that $u(x,0_+) \cong -u(x,0_-)$, and obtain

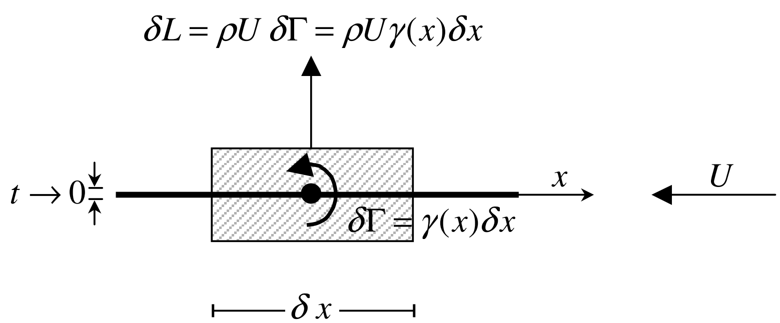
$$u(x,0_\pm) = \mp\frac{\gamma(x)}{2} \tag{2}$$

From Equations (1), and (2) the total lift can be expressed as

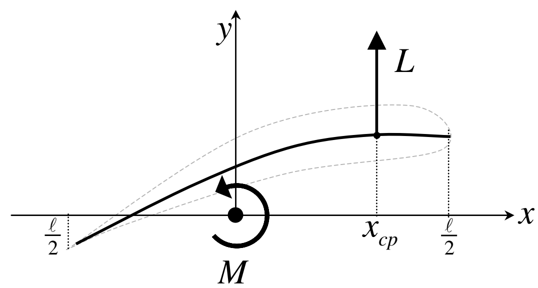
$$L = \rho U \underbrace{\int_{-l/2}^{l/2} \gamma(x) dx}_{=\Gamma} = \rho U \Gamma$$

The same result can be obtained from the Kutta-Joukowski law (for nonlinear foil)

$$\delta L = \rho U \delta\Gamma = \rho U \gamma(x)\delta x \Rightarrow L = \int_{-\ell/2}^{\ell/2} \rho U \gamma(x) \delta x = \rho U \Gamma$$



- **Moment, with respect to mid-chord, due to circulation**



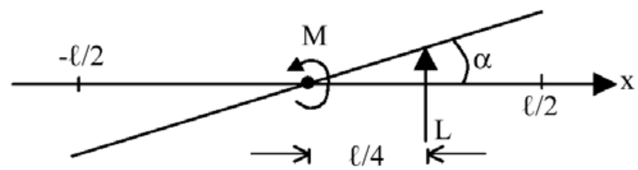
$$
\begin{aligned}
\delta L(x) &= \rho U \gamma(x) \delta x \\
\delta M &= x \delta L(x) = \rho U x \gamma(x) \delta x \Rightarrow \\
M &= \int_{-\ell/2}^{\ell/2} \rho U x \gamma(x) dx \Rightarrow \\
C_M &= \frac{M}{\frac{1}{2}\rho U^2 \ell^2}
\end{aligned}
$$

The center of pressure x_{cp}, can be obtained by

$$
\begin{aligned}
M &= L x_{cp} \Rightarrow \\
x_{cp} &= \frac{M}{L} = \frac{\int_{-\ell/2}^{\ell/2} x \gamma(x) dx}{\int_{-\ell/2}^{\ell/2} \gamma(x) dx}
\end{aligned}
$$

3.17 Simple Closed-Form Solutions for $\int_{-\ell/2}^{\ell/2} \gamma(x)dx$ from Linear Theory

1. Flat plate at angle of attack α, i.e., $\eta = \alpha x$.



Linear lifting theory gives $\gamma(x)$, which can be integrated to give the lift coefficient C_L,

$$
\begin{aligned}
L/\text{span} &= \rho U \int_{-\ell/2}^{\ell/2} \gamma(x)dx = \cdots = \rho U^2 \ell \pi \alpha \Rightarrow \\
C_L &= \frac{L/\text{span}}{\frac{1}{2}\rho U^2 \ell} \Rightarrow \\
C_L &= 2\pi\alpha \text{ (exact nonlinear hydrofoil } C_L = 2\pi \sin\alpha)
\end{aligned}
$$

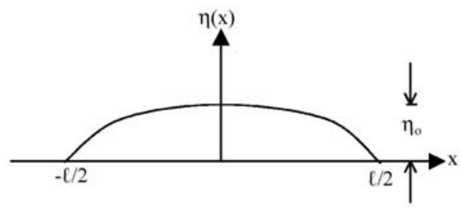
the moment coefficient C_M,

$$
\begin{aligned}
M/\text{span} &= \rho U \int_{-\ell/2}^{\ell/2} x\gamma(x)dx = \cdots = \tfrac{1}{4}\rho U^2 \ell^2 \pi \alpha \Rightarrow \\
C_M &= \frac{M/\text{span}}{\frac{1}{2}\rho U^2 \ell^2} \Rightarrow \\
C_M &= \tfrac{1}{2}\pi\alpha
\end{aligned}
$$

and the center of pressure x_{cp}

$$x_{cp} = \tfrac{1}{4}\ell \;\; \text{i.e., at quarter chord}$$

2. Parabolic camber $\eta = \eta_0\{1 - (\frac{2x}{l})^2\}$, at zero AoA $\alpha = 0$.



Linear lifting theory gives $\gamma(x)$, which can be integrated to give the lift coefficient C_L,

$$\begin{aligned} L/\text{span} &= \rho U \int_{-\ell/2}^{\ell/2} \gamma(x)dx = \cdots = 2\rho U^2 \pi \eta_0 \Rightarrow \\ C_L &= 4\pi \frac{\eta_0}{\ell} \quad \text{, where } \frac{\eta_0}{\ell} \equiv \text{ `camber ratio'} \end{aligned}$$

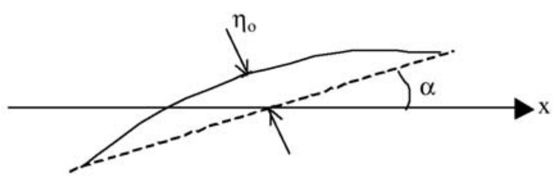
the moment coefficient C_M,

$$\begin{aligned} M/\text{span} &= 0 \text{ (from symmetry) } \Rightarrow \\ C_M &= 0 \end{aligned}$$

and the center of pressure x_{cp}

$$x_{cp} = 0$$

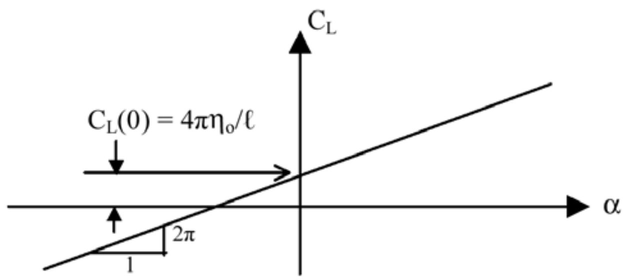
3. Linear superposition: Both AoA and camber $\eta = \alpha x + \eta_0\left(1 - \left(\frac{2x}{\ell}\right)^2\right)$.



$$C_L = C_{L\alpha} + C_{L\eta} = 2\pi\alpha + 4\pi\frac{\eta_0}{\ell}$$

We can also write the previous relation in a more general form

$$C_L(\alpha) = 2\pi\alpha + \underbrace{C_L(\alpha = 0)}_{\equiv 4\pi\frac{\eta_0}{l}}$$



Lift coefficient C_L as a function of the angle of attack α and $\frac{\eta_0}{l}$.

In practice even if the camber is not parabolic, we still make use of the previous relations, i.e., $C_L(\alpha = 0) \cong 4\pi\eta_0/\ell$.
Also note that the angle of attack for any camber is defined as

$$\alpha \equiv \frac{\eta(\ell/2) - \eta(-\ell/2)}{\ell} = \frac{y_U - y_L}{\ell}$$

and η_0 is determined from η^*, where

$$\eta^* = \eta - \alpha x.$$