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2.161 Signal Processing: Continuous and Discrete
Fall 2008

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Massachusetts Institute of Technology

Department of Mechanical Engineering

2.161 *Signal Processing - Continuous and Discrete*
Fall Term 2008

Lecture 15[1]

Reading:

- Proakis & Manolakis, Ch. 7
- Oppenheim, Schafer & Buck. Ch. 10
- Cartinhour, Chs. 6 & 9

1 Frequency Response and Poles and Zeros

As we did for the continuous case, factor the discrete-time transfer functions into as set of poles and zeros;

$$H(z) = \frac{b_0 z^0 + b_1 z^{-2} + \cdots + b_M z^{-M}}{a_0 z^0 + a_1 z^{-2} + \cdots + a_N z^{-N}} = K \frac{\prod_{i=1}^{M}(z - z_i)}{\prod_{i=1}^{N}(z - p_i)}$$

where Z_i are the system *zeros*, the p_i are the system *poles*, and $K = b_0/a_0$ is the overall *gain*. We note, as in the continuous case that the polse and zeros must be either real, or appear in complex conjugate pairs.

As in the continuous case, we can draw a set of vectors from the poles and zeros to a test point in the z-plane, and evaluate $H(z)$ in terms of the lengths and angles of these vectors. In particular, we choose to evaluate $H(\mathrm{e}^{\mathrm{j}\omega})$ on the unit circle,

$$H(\mathrm{e}^{\mathrm{j}\omega}) = K \frac{\prod_{i=1}^{M}(\mathrm{e}^{\mathrm{j}\omega} - z_i)}{\prod_{i=1}^{N}(\mathrm{e}^{\mathrm{j}\omega} - p_i)}$$

and

$$\begin{aligned}
\left|H(\mathrm{e}^{\mathrm{j}\omega})\right| &= K \frac{\prod_{i=1}^{M}\left|\mathrm{e}^{\mathrm{j}\omega} - z_i\right|}{\prod_{i=1}^{N}\left|\mathrm{e}^{\mathrm{j}\omega} - p_i\right|} = K \frac{\prod_{i=1}^{M} q_i}{\prod_{i=1}^{N} r_i} \\
\angle H(\mathrm{e}^{\mathrm{j}\omega}) &= \sum_{i=1}^{M} \angle\left(\mathrm{e}^{\mathrm{j}\omega} - z_i\right) - \sum_{i=1}^{N} \angle\left(\mathrm{e}^{\mathrm{j}\omega} - p_i\right) = \sum_{i=1}^{M}\theta_i - \sum_{i=1}^{N}\phi_i
\end{aligned}$$

where the q_i and θ_i are the lengths and angles of the vectors from the zeros to the point $z = \mathrm{e}^{\mathrm{j}\omega}$, the r_i and ϕ_i are the lengths and angles of the vectors from the poles to the point $z = \mathrm{e}^{\mathrm{j}\omega}$, as shown below:

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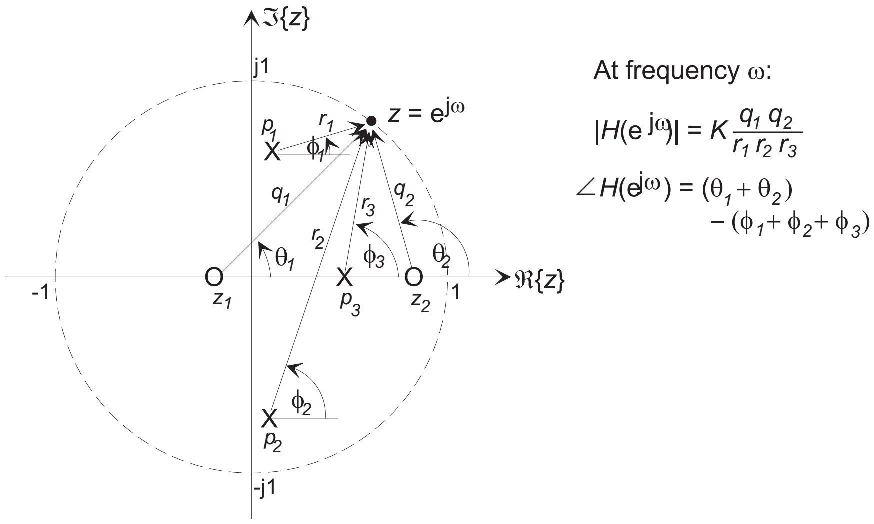


We can interpret the effect of pole and zero locations on the frequency response as follows:

(a) A pole (or conjugate pole pair) on the unit circle will cause $\left|H(e^{j\omega})\right|$ to become infinite at frequency ω.

(b) A zero (or conjugate zero pair) on the unit circle will cause $\left|H(e^{j\omega})\right|$ to become zero at frequency ω.

(c) Poles near the unit circle will cause a peak in $\left|H(e^{j\omega})\right|$ in the neighborhood of those poles.

(c) Zeros near the unit circle will cause a dip, or notch, in $\left|H(e^{j\omega})\right|$ in the neighborhood of those zeros.

(d) Poles and zeros at the origin $z = 0$ have no effect upon $\left|H(e^{j\omega})\right|$, but add a frequency dependent linear phase taper ($-\omega$ for a pole, $+\omega$ for a zero), which is equivalent to shift.

(e) A pole or zero at $z = 1$ forces $\left|H(e^{j\omega})\right|$ to be infinite (pole) or 0 (zero) at $\omega = 0$.

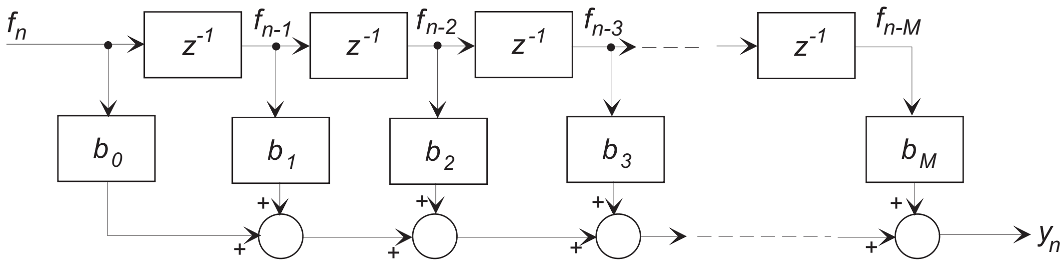
(f) A pole or zero at $z = -1$ forces $\left|H(e^{j\omega})\right|$ to be infinite (pole) or 0 (zero) at $\omega = \pi$, (the Nyquist frequency).

2 FIR Low-Pass Filter Design

The FIR (finite impulse response) filter is an *all-zero* system with a difference equation

$$y_n = \sum_{k=0}^{M} b_k f_{n-k}$$

which is clearly a convolution of the input sequence with an impulse response $\{h_k\} = \{b_k\}$, for $k = 0, \cdots, M$. A direct-form causal implementation is



The transfer function is

$$H(z) = \sum_{k=0}^{M} b_k z^{-k} = \frac{1}{z^M} \sum_{k=0}^{M} b_k z^{M-k}$$

and the frequency response is

$$H(\,\mathrm{e}^{\mathrm{j}\,\omega}) = \sum_{k=0}^{M} b_k \,\mathrm{e}^{-\mathrm{j}\,k\omega}.$$

Note that there are $M+1$ terms in the impulse response but the order of the polynomials is M.

■ Example 1

Find the frequency response $H(\,\mathrm{e}^{\mathrm{j}\,\omega})$ for a simple three-point moving average filter:

$$y_n = \frac{1}{3}\left(f_n + f_{n-1} + f_{n-2}\right).$$

Solution:

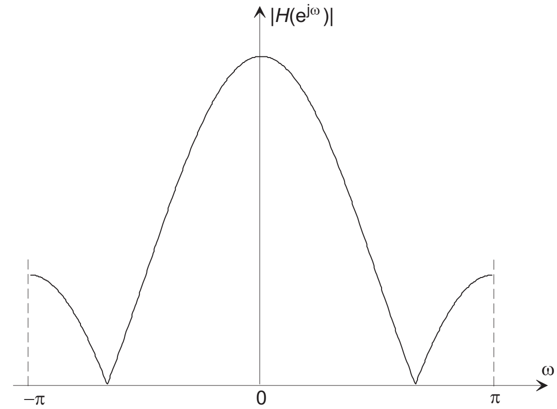
$$H(z) = \frac{1}{3} z^0 + \frac{1}{3} z^{-1} + \frac{1}{3} z^{-2}$$

so that

$$\begin{aligned} H(\,\mathrm{e}^{\mathrm{j}\,\omega}) &= \frac{1}{3}\left(1 + \,\mathrm{e}^{-\mathrm{j}\,\omega} + \,\mathrm{e}^{-2\mathrm{j}\,\omega}\right) \\ &= \frac{1}{3}\,\mathrm{e}^{-\mathrm{j}\,\omega}\left(\,\mathrm{e}^{-\mathrm{j}\,\omega} + 1 + \,\mathrm{e}^{\mathrm{j}\,\omega}\right) \\ &= \frac{1}{3}\left(1 + 2\cos(\omega)\right)\,\mathrm{e}^{-\mathrm{j}\,\omega} \end{aligned}$$

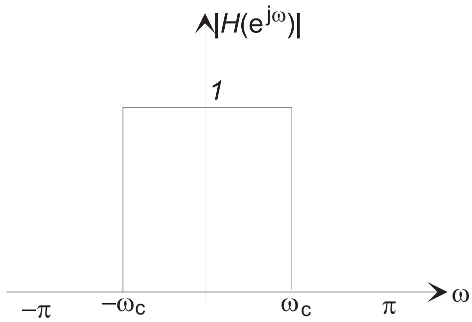
and

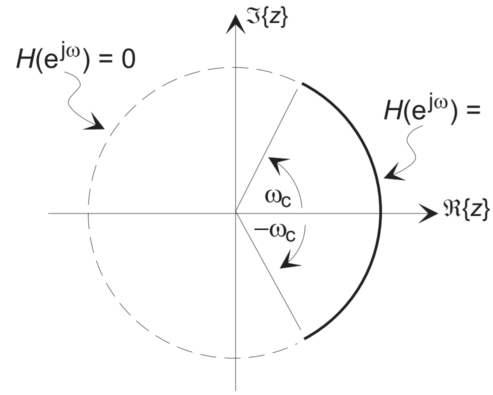
$$\begin{aligned} \left|H(\,\mathrm{e}^{\mathrm{j}\,\omega})\right| &= \frac{1}{3}\left(1 + 2\cos(\omega)\right) \\ \angle H(\,\mathrm{e}^{\mathrm{j}\,\omega}) &= -\omega. \end{aligned}$$



The ideal FIR low-pass filter has a response

$$H(\mathrm{e}^{\mathrm{j}\,\omega}) = \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$





The impulse response $h_n = \mathcal{Z}^{-1}\{H(z)\}$, and although we are not given $H(z)$ explicitly, we can use the formal definition of the inverse z-transform (Lecture 14) as a contour integral in the z-plane,

$$\mathcal{Z}^{-1}\{H(z)\} = \frac{1}{2\pi\mathrm{j}} \oint_{-\infty}^{\infty} H(z) z^{n-1}\,\mathrm{d}z$$

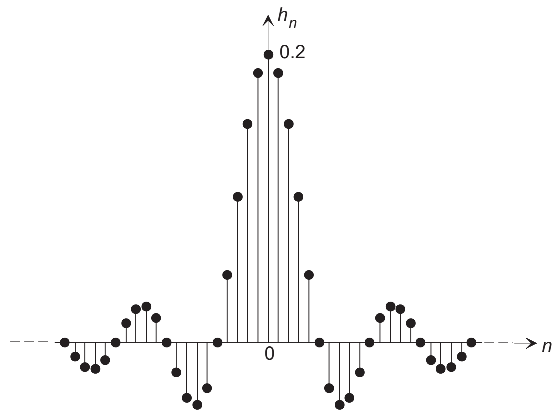
where the path is a ccw contour enclosing all of the poles of $H(z)$, and for a stable filter choose the contour as the unit-circle. Let $z = \mathrm{e}^{\mathrm{j}\,\omega}$, so that $\mathrm{d}z = \mathrm{j}\,\mathrm{e}^{\mathrm{j}\,\omega}\,\mathrm{d}\omega$, and

$$h_n = \mathcal{Z}^{-1}\{H(z)\} = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1.\,\mathrm{e}^{\mathrm{j}\,n\omega}\,\mathrm{d}\omega = \frac{\omega_c}{\pi}\left(\frac{\sin(\omega_c n)}{\omega_c n}\right)$$

The impulse response of the FIR ideal low-pass filter is therefore

$$\boxed{h_n = \frac{\omega_c}{\pi}\left(\frac{\sin(\omega_c n)}{\omega_c n}\right)}$$

The following figure shows the central region of the impulse response of an ideal FIR filter with $\omega_c = 0.2\pi$:



It is obvious that this impulse response has two problems:

(a) It is infinite in extent, and

(b) It is non-causal.

> To produce a causal, finite length filter
> **(a)** Truncate $\{h_n\}$ to include $M+1$ central points ($M+1$ odd), that is select the points $-M/2 \le n \le M/2$. Let this truncated filter be designated $\hat{H}(z)$.
>
> **(b)** Shift the truncated impulse response $\{\hat{h}_n\}$ to the right by $M/2$ to form a causal sequence $\{h'_n\}$, where $h'_n = \hat{h}_{(M/2-n)}$, for $n = 0, \ldots M$.
> Take $\{h'_n\}$ as the FIR causal approximation to the ideal low-pass filter.

Then

$$H'(z) = z^{(M-1)/2}\hat{H}(z),$$

that is the response is delayed by $(M-1)/2$ samples. The frequency response is

$$H'(\mathrm{e}^{\mathrm{j}\,\omega}) = \mathrm{e}^{\mathrm{j}\,(M-1)\omega/2}\hat{H}(\mathrm{e}^{\mathrm{j}\,\omega}),$$

and because $\hat{H}(\mathrm{e}^{\mathrm{j}\,\omega})$ is real

$$\begin{aligned} \left|H'(\mathrm{e}^{\mathrm{j}\,\omega})\right| &= \left|\hat{H}(\mathrm{e}^{\mathrm{j}\,\omega})\right| \\ \angle H'(\mathrm{e}^{\mathrm{j}\,\omega}) &= (M-1)\omega/2. \end{aligned}$$

■ Example 2

Design a five point causal FIR low-pass filter with a cut-off frequency $\omega_c = 0.4\pi$.

Solution: The ideal filter has an impulse response

$$h_n = \frac{\omega_c}{\pi}\left(\frac{\sin(\omega_c n)}{\omega_c n}\right) = \frac{1}{2}\left(\frac{\sin(\pi n/2)}{\pi n/2}\right)$$

Select $M+1=5$, and select the five central components:

$n:$	-2	-1	0	1	2
$h'_n:$	0.0935	0.3027	0.4	0.3027	0.0935

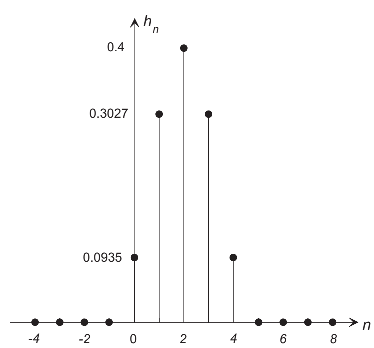
Shift to the right by $M/2 = 2$, and form the causal impulse response $\{\hat{h}_n\}$:

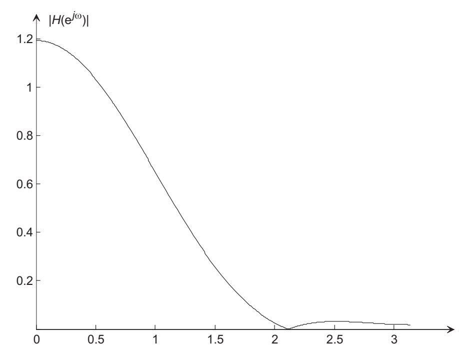
$n:$	0	1	2	3	4
$\hat{h}_n:$	0.0935	0.3027	0.4	0.3027	0.0935

with difference equation

$$y_n = 0.0935 f_n + 0.3027 f_{n-1} + 0.4 f_{n-2} + 0.3027 f_{n-3} + 0.0935 f_{n-4}$$

The causal impulse response and the frequency response magnitude are shown below:





2.1 The Effect of Truncation and Shifting

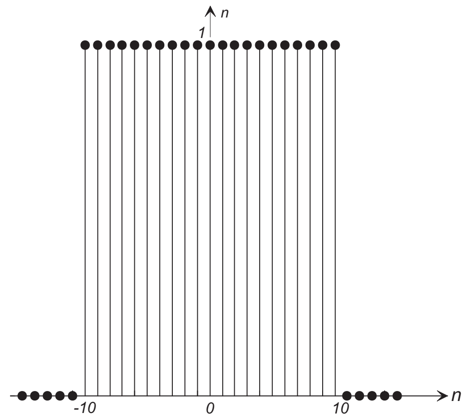
(a) The Effect of Truncation The selection of the $M+1$ central components of the non-causal impulse response $\{h_n\}$ can be written as a product

$$\{h'_n\} = \{h_n r_n\}$$

where $\{r_n\}$ is an even *rectangular window function*

$$r_n = \begin{cases} 1 & |n| \le M/2 \\ 0 & \text{otherwise.} \end{cases}$$

The following figure shows $\{r_n\}$ for $M = 20$.



The truncated frequency response is therefore

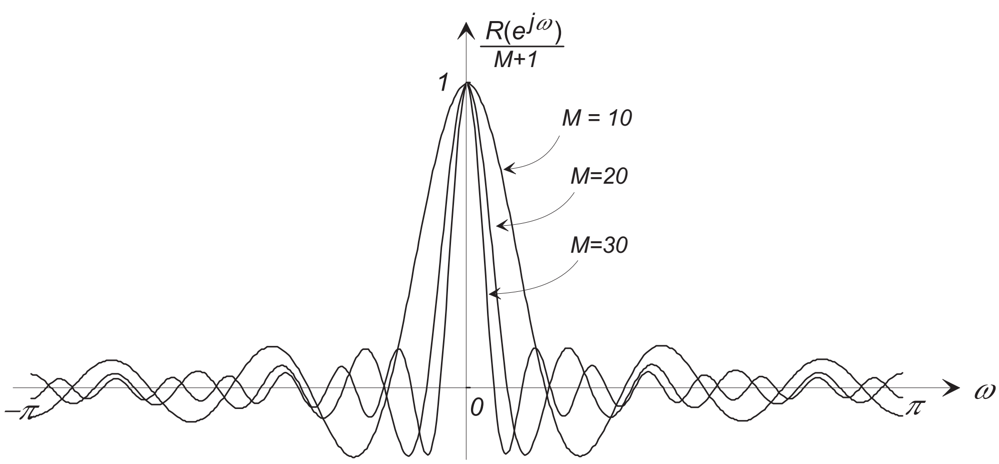
$$H'(\mathrm{e}^{\mathrm{j}\,\omega}) = \frac{1}{2\pi} H(\mathrm{e}^{\mathrm{j}\,\omega}) \otimes R(\mathrm{e}^{\mathrm{j}\,\omega}).$$

For the window function

$$\begin{aligned} R(z) &= \sum_{k=-M/2}^{M/2} z^{-1} \\ R(\mathrm{e}^{\mathrm{j}\,\omega}) &= \sum_{k=-M/2}^{M/2} \mathrm{e}^{-\mathrm{j}\,k\omega} = 1 + \sum_{k=1}^{M/2} 2\cos(k\omega) = D_{M/2}(\omega) \end{aligned}$$

$D_{M/2}(\omega)$ is known as the *Dirichlet kernel*, and is found in the study of truncated Fourier series and convolution of periodic functions. It is easy to show (using the sum of a finite geometric series) that

$$R(\mathrm{e}^{\mathrm{j}\,\omega}) = D_{M/2}(\omega) = \frac{\sin((M+1)\omega/2)}{\sin(\omega/2)}$$



Notice (1) The width of the main lobe decreases with M, (2) the side lobes do not decay to zero as ω increases.

Aside: The formal definition of the z-transform of the product of two sequences is given by the z-plane contour integral

$$F(z) = \mathcal{Z}\{x_n y_n\} = \frac{1}{2\pi \mathrm{j}} \oint X(\nu) Y\left(\frac{z}{\nu}\right) \nu^{-1} \, \mathrm{d}\nu$$

where $X(z) = \mathcal{Z}\{x_n\}$, $Y(z) = \mathcal{Z}\{y_n\}$, and the contour lies in the ROC of both $X(z)$ and $Y(z)$. In particular if the unit-circle lies within the ROC of both sequences, choose the unit-circle as the contour, $\nu = \mathrm{e}^{\mathrm{j}\,\omega}$, then

$$F(\mathrm{e}^{\mathrm{j}\,\omega_o}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\mathrm{e}^{\mathrm{j}\,\omega}) Y(\mathrm{e}^{\mathrm{j}\,(\omega_0 - \omega)}) \, \mathrm{d}\omega$$

which is the convolution of $X(\mathrm{e}^{\mathrm{j}\,\omega})$ and $Y(\mathrm{e}^{\mathrm{j}\,\omega})$.

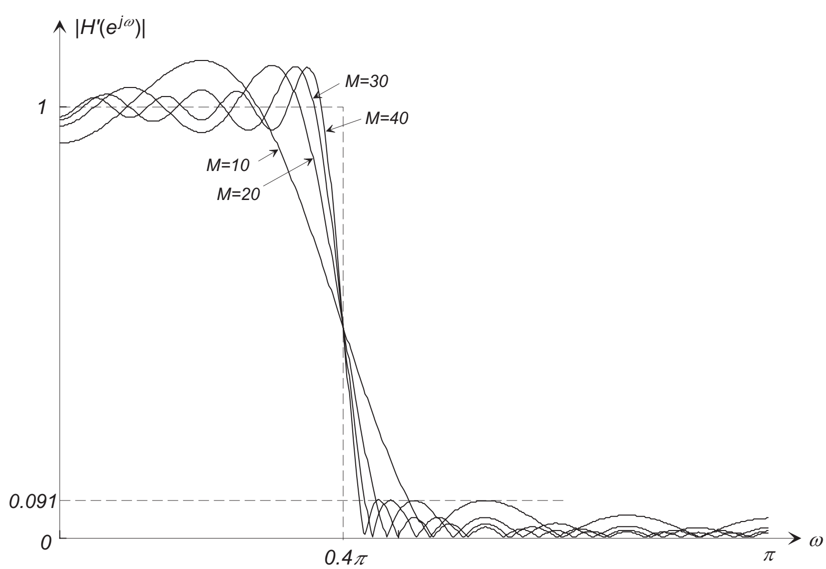
The frequency response of the truncated filter is

$$\begin{aligned} H'(\mathrm{e}^{\mathrm{j}\,\omega}) &= \frac{1}{2\pi} H(\mathrm{e}^{\mathrm{j}\,\omega}) \otimes R(\mathrm{e}^{\mathrm{j}\,\omega}) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\mathrm{e}^{\mathrm{j}\,\nu}) R(\mathrm{e}^{\mathrm{j}\,(\nu - \omega)}) \, \mathrm{d}\nu \end{aligned}$$

or

$$\boxed{H'(\mathrm{e}^{\mathrm{j}\,\omega}) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} D_{M/2}(\nu - \omega) \, \mathrm{d}\nu.}$$

which is shown below for filter with $\omega_c = 0.4\pi$ and lengths $M + 1 = 11, 21, 31$, and 41:



In general:

- The amplitude of the ripple in the pass-band does not decrease with the filter order M.
- Similarly, the stop-band attenuation is relatively unaffected by the filter order, and the amplitude of the first side-lobe is 0.091, so that the truncated filter has a stop-band attenuation of -21 dB.
- The width of the transition-band decreases with increasing M.

(b) The Effect of the Right-Shift to Form a Causal Filter The truncated non-causal impulse response $\{h'\}$ is even and real, so that $H'(e^{j\omega})$ is also real and even, that is $\angle H'(e^{j\omega}) = 0$.

The right-shift of h' by $M/2$ samples to force causality imposes

$$\hat{H}(z) = z^{-M/2} H'(z)$$

and therefore

$$\hat{H}(e^{j\omega}) = e^{-j\omega M/2} H'(e^{j\omega}).$$

The phase response of the filter is

$$\angle H'(e^{j\omega}) = -(M/2)\omega$$

which is a linear phase taper (lag).

The effect of the right-shift on the impulse response of the ideal filter is to impose a phase lag that is proportional to frequency, with a slope of $-M/2$.