2.161 Signal Processing: Continuous and Discrete
Fall 2008

Lecture 5[1]

Reading:

- Class handout: *The Laplace Transform.*

- Class handout: *Understanding Poles and Zeros.*

- Class handout: *Sinusoidal Frequency Response of Linear Systems.*

1   The One-Sided Laplace Transform

Consider a causal waveform $x(t)$, such as the unit-step function $u_s(t)$, for which the Fourier integral

$$\int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt = \int_{0^-}^{\infty} x(t)e^{-j\Omega t}dt$$

does not converge. Clearly, for $u_s(t)$, the Fourier integral

$$U_s(j\Omega) = \int_{0^-}^{\infty} 1.e^{-j\Omega t}dt$$

does not converge.

Now consider a modified function $\widehat{x}(t) = x(t)w(t)$ where $w(t)$ is a *weighting* function with the property $\lim_{t\to\infty} w(t) = 0$, chosen to ensure convergence, so that

$$\widehat{X}(j\Omega, w) = \int_{0^-}^{\infty} x(t)w(t)e^{-j\Omega t}dt$$

may be considered an approximation to $X(j\Omega)$.

In particular, consider $w(t) = e^{-\sigma t}$ for real σ, and note that as $\sigma \to 0$, $x(t)w(t) \to x(t)$, so that the Fourier transform is

$$\mathcal{F}\left\{x(t)e^{-\sigma t}\right\} = X(j\Omega, \sigma) = \int_{0^-}^{\infty} x(t)e^{-\sigma t}e^{-j\Omega t}dt = \int_{0^-}^{\infty} x(t)e^{-(\sigma+j\Omega)t}dt.$$

If we define a complex variable $s = \sigma + j\Omega$ we can write

$$\mathcal{L}\left\{x(t)\right\} = X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$$

which defines the *one-sided* Laplace transform. (See the handout for the definition of the two-sided transform).

The Laplace transform may be considered as an extension of the Fourier transform
(for causal functions) that includes an exponential weighting term to extend the range
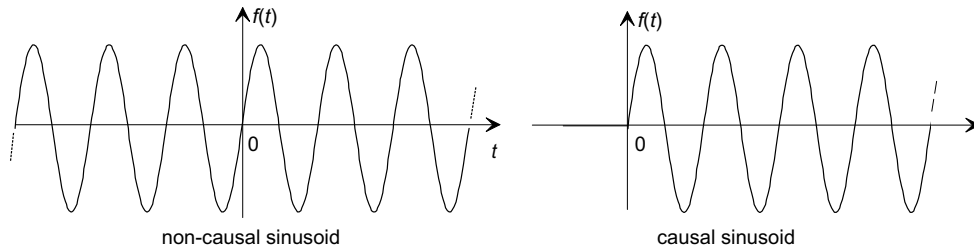of functions for which the integral converges.

Note that for causal waveforms

$$X(j\Omega) = X(s)|_{s=j\Omega}$$

and if $x(t)$ is non-causal

$$X(j\Omega) \neq X(s)|_{s=j\Omega},$$

for example $\mathcal{F}\{\sin(\Omega_0 t)\} \neq \mathcal{L}\{\sin(\Omega_0 t)\}|_{s=j\Omega}$, since the Laplace transform assumes $x(t) \equiv 0$
for $t < 0^-$.

non-causal sinusoid   causal sinusoid

■ Example 1

The following are some simple examples of Laplace transforms:

$$\mathcal{L}\{u_s(t)\} = \int_{0^-}^{\infty} 1.e^{-st}dt = \frac{1}{s}$$

$$\mathcal{L}\{\delta(t)\} = \int_{0^-}^{\infty} \delta(t)e^{-st}dt = 1$$

$$\mathcal{L}\{e^{-at}\} = \int_{0^-}^{\infty} e^{-(s+a)t}dt = \frac{1}{s+a}$$

1.1   The Derivative Property of the Laplace Transform:

If a function $x(t)$ has a Laplace transform $X(s)$, the Laplace transform of the derivative of
$x(t)$ is

$$\mathcal{L}\left\{\frac{dx}{dt}\right\} = sX(s) - x(0).$$

Using integration by parts

$$\mathcal{L}\left\{\frac{dx}{dt}\right\} = \int_{0^-}^{\infty} \frac{dx}{dt}e^{-st}dt$$

$$= \left| x(t)e^{-st}\right|_{0^-}^{\infty} + \int_{0^-}^{\infty} sx(t)e^{-st}dt$$

$$= sX(s) - x(0).$$

5–2

This procedure may be repeated to find the Laplace transform of higher order derivatives, for example the Laplace transform of the second derivative is

$$
\begin{aligned}
\mathcal{L}\left\{\frac{d^2 x}{dt^2}\right\} &= s\left[s\mathcal{L}\left\{x(t)\right\} - x(0)\right] - \frac{dx}{dt}\bigg|_{t=0} \\
&= s^2 X(s) - sx(0) - \frac{dx}{dt}\bigg|_{t=0}
\end{aligned}
$$

which may be generalized to

$$
\mathcal{L}\left\{\frac{d^n x}{dt^n}\right\} = s^n X(s) - \sum_{i=1}^{n} s^{n-i}\left(\frac{d^{i-1} x}{dt^{i-1}}\bigg|_{t=0}\right)
$$

for the n derivative of $x(t)$.

2 The Transfer Function

The use of the derivative property of the Laplace transform generates a direct algebraic solution method for determining the response of a system described by a linear input/output differential equation. Consider an nth order linear system, completely relaxed at time $t = 0$, and described by

$$
\begin{aligned}
a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = \\
b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1} u}{dt^{m-1}} + \ldots + b_1 \frac{du}{dt} + b_0 u.
\end{aligned}
$$

In addition assume that the input function $u(t)$, and all of its derivatives are zero at time $t = 0$, and that any discontinuities occur at time $t = 0^+$. Under these conditions the Laplace transforms of the derivatives of both the input and output simplify to

$$
\mathcal{L}\left\{\frac{d^n y}{dt^n}\right\} = s^n Y(s), \quad \text{and} \quad \mathcal{L}\left\{\frac{d^n u}{dt^n}\right\} = s^n U(s)
$$

so that if the Laplace transform of both sides is taken

$$
\begin{aligned}
\left\{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0\right\} Y(s) = \\
\left\{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0\right\} U(s)
\end{aligned}
$$

which has had the effect of reducing the original differential equation into an algebraic equation in the complex variable s. This equation may be rewritten to define the Laplace transform of the output:

$$
\begin{aligned}
Y(s) &= \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} U(s) \\
&= H(s)U(s)
\end{aligned}
$$

The Laplace transform generalizes the definition of the transfer function to a complete input/output description of the system for any input $u(t)$ that has a Laplace transform.

The system response $y(t) = \mathcal{L}^{-1}\{Y(s)\}$ may be found by decomposing the expression for $Y(s) = U(s)H(s)$ into a sum of recognizable components using the method of partial fractions as described above, and using tables of Laplace transform pairs to find the component time domain responses. To summarize, the Laplace transform method for determining the response of a system to an input $u(t)$ consists of the following steps:

(1) If the transfer function is not available it may be computed by taking the Laplace transform of the differential equation and solving the resulting algebraic equation for $Y(s)$.

(2) Take the Laplace transform of the input.

(3) Form the product $Y(s) = H(s)U(s)$.

(4) Find $y(t)$ by using the method of partial fractions to compute the inverse Laplace transform of $Y(s)$.
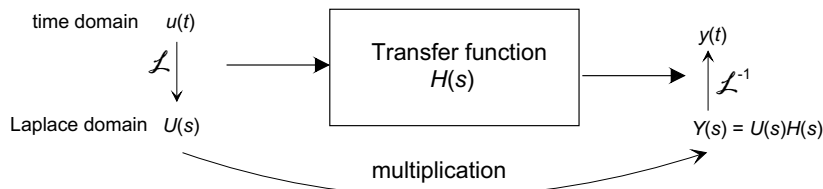
■ Example 2

Determine the transfer function of the first-order RC filter:
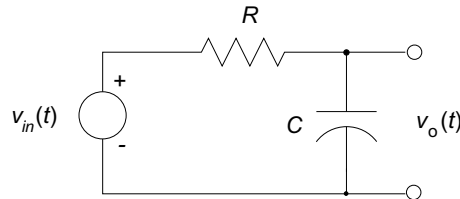
The differential equation relating $v_o(t)$ to $v_{in}(t)$ is

$$RC\frac{dv_0}{dt} + v_o = v_{in}(t)$$

and taking the Laplace transform of both sides gives

$$(RCs + 1)V_o(s) = V_{in}(s)$$

from which

$$H(s) = \frac{V_o(s)}{V_{in}(s)} = \frac{1}{RCs + 1}$$

2.1 The Transfer function and the Sinusoidal Frequency Response

We have seen that

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$$

and

$$H(j\Omega) = \frac{b_m (j\Omega)^m + b_{m-1} (j\Omega)^{m-1} + \ldots + b_1 (j\Omega) + b_0}{a_n (j\Omega)^n + a_{n-1} (j\Omega)^{n-1} + \ldots + a_1 (j\Omega) + a_0}$$

so that

$$H(j\Omega) = H(s)|_{s=j\Omega}$$

3 Poles and Zeros of the Transfer Function

The transfer function provides a basis for determining important system response characteristics without solving the complete differential equation. As defined, the transfer function is a rational function in the complex variable $s = \sigma + j\Omega$, that is

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$$

It is often convenient to factor the polynomials in the numerator and denominator, and to write the transfer function in terms of those factors:

$$H(s) = \frac{N(s)}{D(s)} = K \frac{(s - z_1)(s - z_2) \ldots (s - z_{m-1})(s - z_m)}{(s - p_1)(s - p_2) \ldots (s - p_{n-1})(s - p_n)},$$

where the numerator and denominator polynomials, $N(s)$ and $D(s)$, have real coefficients defined by the system's differential equation and $K = b_m/a_n$. The z_i's are the roots of the equation

$$N(s) = 0,$$

and are defined to be the system *zeros*, and the p_i's are the roots of the equation

$$D(s) = 0,$$

and are defined to be the system *poles*. Clearly when $s = z_i$ the numerator $N(s) = 0$ and the transfer function vanishes, that is

$$\lim_{s \to z_i} H(s) = 0.$$

and similarly when $s = p_i$ the denominator polynomial $D(s) = 0$ and the value of the transfer function becomes unbounded,

$$\lim_{s \to p_i} H(s) = \infty.$$

All of the coefficients of polynomials $N(s)$ and $D(s)$ are real, therefore the poles and zeros must be either purely real, or appear in complex conjugate pairs. In general for the poles, either $p_i = \sigma_i$, or else $p_i, p_{i+1} = \sigma_i \pm j\Omega_i$. The existence of a single complex pole without a corresponding conjugate pole would generate complex coefficients in the polynomial $D(s)$. Similarly, the system zeros are either real or appear in complex conjugate pairs.

3.1 The Pole-Zero Plot

A system is characterized by its poles and zeros in the sense that they allow reconstruction of the input/output differential equation. In general, the poles and zeros of a transfer function may be complex, and the system dynamics may be represented graphically by plotting their locations on the complex s-plane, whose axes represent the real and imaginary parts of the complex variable s. Such plots are known as *pole-zero plots*. It is usual to mark a zero location by a circle ($\circ$) and a pole location a cross ($\times$). The location of the poles and zeros provide qualitative insights into the response characteristics of a system. Many computer programs are available to determine the poles and zeros of a system. The figure below is an example of a pole-zero plot for a third-order system with a single real zero, a real pole and a complex conjugate pole pair, that is;

$$H(s) = \frac{(3s+6)}{(s^3 + 3s^2 + 7s + 5)} = 3\frac{(s-(-2))}{(s-(-1))(s-(-1-2j))(s-(-1+2j))}$$
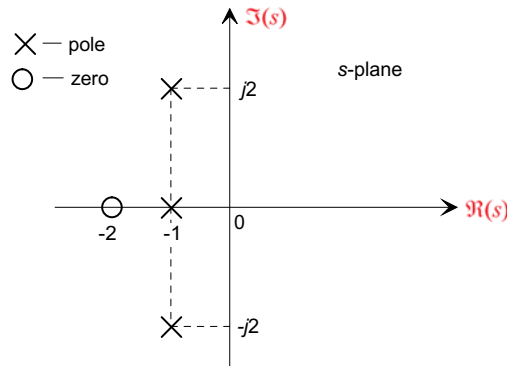
Note that the pole-zero plot characterizes the system, except for the overall gain constant K.

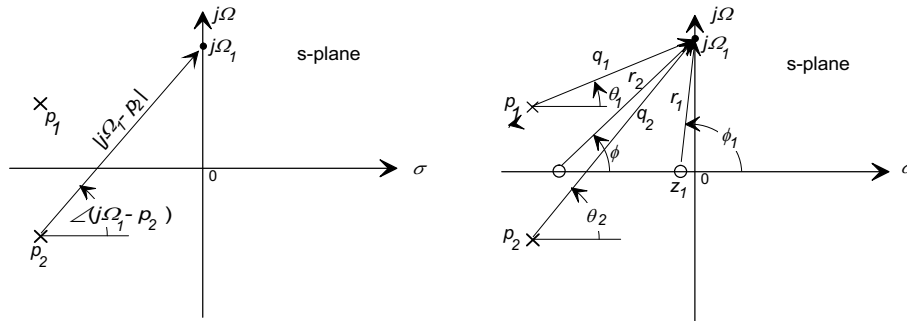# 4 Frequency Response and the Pole-Zero Plot

The frequency response may be written in terms of the system poles and zeros by substituting directly into the factored form of the transfer function:

$$H(j\Omega) = K\frac{(j\Omega - z_1)(j\Omega - z_2)\ldots(j\Omega - z_{m-1})(j\Omega - z_m)}{(j\Omega - p_1)(j\Omega - p_2)\ldots(j\Omega - p_{n-1})(j\Omega - p_n)}.$$

Because the frequency response is the transfer function evaluated on the imaginary axis of the s-plane, that is when $s = j\Omega$, the graphical method for evaluating the transfer function may be applied directly to the frequency response. Each of the vectors from the n system poles to a test point $s = j\Omega$ has a magnitude and an angle:

$$\begin{aligned}
|j\Omega - p_i| &= \sqrt{\sigma_i^2 + (\Omega - \Omega_i)^2}, \\
\angle(s - p_i) &= \tan^{-1}\left(\frac{\Omega - \Omega_i}{-\sigma_i}\right),
\end{aligned}$$

as shown above, with similar expressions for the vectors from the m zeros. The magnitude and phase angle of the complete frequency response may then be written in terms of the magnitudes and angles of these component vectors

$$|H(j\Omega)| = K\frac{\prod_{i=1}^{m}|(j\Omega - z_i)|}{\prod_{i=1}^{n}|(j\Omega - p_i)|}$$

$$\angle H(j\Omega) = \sum_{i=1}^{m}\angle(j\Omega - z_i) - \sum_{i=1}^{n}\angle(j\Omega - p_i).$$

If the vector from the pole p_i to the point $s = j\Omega$ has length q_i and an angle θ_i from the horizontal, and the vector from the zero z_i to the point $j\Omega$ has a length r_i and an angle ϕ_i, the value of the frequency response at the point $j\Omega$ is

$$|H(j\Omega)| = K\frac{r_1 \ldots r_m}{q_1 \ldots q_n} \tag{1}$$

$$\angle H(j\Omega) = (\phi_1 + \ldots + \phi_m) - (\theta_1 + \ldots + \theta_n) \tag{2}$$

The graphical method can be very useful for deriving a qualitative picture of a system frequency response. For example, consider the sinusoidal response of a first-order system with a pole on the real axis at $s = -1/\tau$ as shown below. Even though the gain constant K cannot be determined from the pole-zero plot, the following observations may be made directly by noting the behavior of the magnitude and angle of the vector from the pole to the imaginary axis as the input frequency is varied:

(a)

(b)

5–7
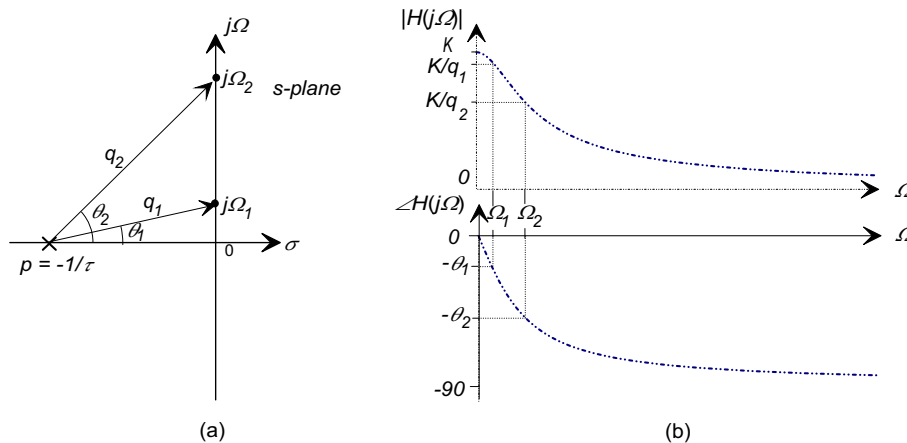
1. At low frequencies the gain approaches a finite value, and the phase angle has a small but finite lag.

2. As the input frequency is increased the gain decreases (because the length of the vector increases), and the phase lag also increases (the angle of the vector becomes larger).

3. At very high input frequencies the gain approaches zero, and the phase angle approaches $\pi/2$.

As a second example consider a second-order system, with the damping ratio chosen so that the pair of complex conjugate poles are located close to the imaginary axis as shown below. In this case there are a pair of vectors connecting the two poles to the imaginary axis, and the following conclusions may be drawn by noting how the lengths and angles of the vectors change as the test frequency moves up the imaginary axis:

(a)   (b)
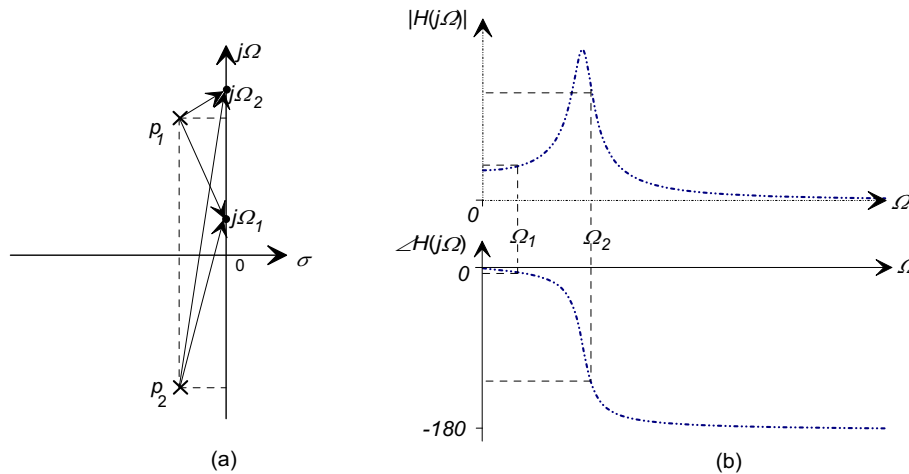
1. At low frequencies there is a finite (but undetermined) gain and a small but finite phase lag associated with the system.

2. As the input frequency is increased and the test point on the imaginary axis approaches the pole, one of the vectors (associated with the pole in the second quadrant) decreases in length and at some point reaches a minimum. There is an increase in the value of the magnitude function over a range of frequencies close to the pole.

3. At very high frequencies, the lengths of both vectors tend to infinity, and the magnitude of the frequency response tends to zero, while the phase approaches an angle of π radians because the angle of each vector approaches $\pi/2$.

   The following generalizations may be made about the sinusoidal frequency response of a linear system, based upon the geometric interpretation of the pole-zero plot:

1. If a system has an excess of poles over the number of zeros $(n > m)$ the magnitude of the frequency response tends to zero as the frequency becomes large. Similarly, if a system has an excess of zeros $(n < m)$ the gain increases without bound as the

frequency of the input increases. (This cannot happen in physical energetic systems because it implies an infinite power gain through the system.) If $n = m$ the system gain becomes constant at high frequencies.

2. If a system has a pair of complex conjugate poles close to the imaginary axis, the magnitude of the frequency response has a "peak", or resonance, at frequencies in the proximity of the pole. If the pole pair lies directly upon the imaginary axis, the system exhibits an infinite gain at that frequency.

3. If a system has a pair of complex conjugate zeros close to the imaginary axis, the frequency response has a "dip" or "notch" in its magnitude function at frequencies in the vicinity of the zero. Should the pair of zeros lie directly upon the imaginary axis, the response is identically zero at the frequency of the zero, and the system does not respond at all to sinusoidal excitation at that frequency.

4. A pole at the origin of the s-plane (corresponding to a pure integration term in the transfer function) implies an infinite gain at zero frequency.

5. Similarly a zero at the origin of the s-plane (corresponding to a pure differentiation) implies a zero gain for the system at zero frequency.