NONLINEAR MECHANICAL SYSTEMS

LAGRANGIAN AND HAMILTONIAN FORMULATIONS

Lagrangian formulation

$$E_k^*(f,q) = \frac{1}{2}\, f^t\, I(q)\, f$$

q generalized coordinates (displacement)

f generalized velocity (flow)

$E_k^*(f,q)$ kinetic co-energy

I(q) a configuration-dependent inertia tensor (matrix)

Note:

kinetic co-energy is a quadratic form in flow (generalized velocity).

(Euler-)Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial E_k^*}{\partial f}\right) - \frac{\partial E_k^*}{\partial q} = e$$

General form:

$$I(q)\frac{df}{dt} - C(f,q) = e$$

e generalized force (effort)

C(f,q) contains coriolis and centrifugal "forces"

Explicit state-determined form:

$$\frac{dq}{dt} = f$$

$$\frac{df}{dt} = I(q)^{-1}(e + C(f,q))$$

Note:

The inertia tensor (matrix) must be inverted to find a state-determined form.

Lagrange's equation may include conservative generalized forces.

$$e = e_{conservative} + e_{non\text{-}conservative}$$

$$e_{conservative} = -\frac{\partial Ep(q)}{\partial q}$$

$E_p(q)$ potential energy function

$$\frac{d}{dt}\left(\frac{\partial Ek^*}{\partial f}\right) - \frac{\partial Ek^*}{\partial q} + \frac{\partial Ep(q)}{\partial q} = e_{non\text{-}conservative}$$

Potential energy is not a function of generalized velocity.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial f}\right) - \frac{\partial L}{\partial q} = e_{non\text{-}conservative}$$

L(f,q) is the Lagrangian state function

$$L(f,q) = E_k^*(f,q) - E_p(q)$$

In the usual notation, $E_k^*(f,q)$ is written as T(f,q) and $E_p(q)$ is written as V(q), hence

$$L(f,q) = T(f,q) - V(q)$$

i.e.,

$$L = T - V$$

Hamiltonian formulation

Interaction between a capacitor and an inertia is the archetypal Hamiltonian system:

$dp/dt = -\partial H(p,q)/\partial q$

$dq/dt = \partial H(p,q)/\partial p$

q generalized coordinates

p generalized momenta

H(p,q) is the Hamiltonian state function.

$H(p,q) = E_k(p,q) + Ep(q)$

In this case H(p,q) is equal to the total energy in the system.

Lagrangian and Hamiltonian forms are related by a Legendre transformation.

Lagrange used kinetic co-energy

Hamilton used kinetic energy

$$H(p,q) = p^t f - L(f,q)$$

$$L(f,q) = p^t f - H(p,q)$$

Differentiate with respect to generalized momentum.

$$\partial L(f,q)/\partial p = 0 = f - \partial H(p,q)/\partial p$$

thus

$$dq/dt = \partial H(p,q)/\partial p$$

Generalized momentum is the gradient of kinetic co-energy with respect to flow (velocity)

$$p = \partial E_k^*(f,q)/\partial f = \partial L(f,q)/\partial f$$

Lagrange's equation:

$$dp/dt - \partial L(f,q)/\partial q = e$$

$$\partial L(f,q)/\partial q = -\partial H(p,q)/\partial q$$

thus

$$dp/dt = -\partial H(p,q)/\partial q + e$$

These are Hamilton's equations

$$dq/dt = \partial H(p,q)/\partial p$$

$$dp/dt = -\partial H(p,q)/\partial q + e$$

EXAMPLE: MULTIPLE DEGREE OF FREEDOM MECHANISM

Generalized coordinates: q

Generalized momentum

$$p = \partial E_k^*(f,q)/\partial f = I(q)\, f$$

Kinetic co-energy is a positive definite quadratic form,.

The inertia tensor is symmetric and real.

Kinetic energy

$$E_k(p,q) = p^t f - E_k^*(f,q) = p^t f - \frac{1}{2}\, f^t I(q) f$$

$$E_k(p,q) = p^t I(q)^{-1} p - \frac{1}{2}\, p^t I(q)^{-1} I(q) I(q)^{-1} p$$

$$E_k(p,q) = \frac{1}{2}\, p^t I(q)^{-1} p$$

Note:

Kinetic energy is always a quadratic form in generalized momentum.

Hamilton's equations

$$dp/dt = -\partial[\tfrac{1}{2}\ p^t I(q)^{-1} p]/\partial q + e$$

$$dq/dt = I(q)^{-1} p$$

Note:

The Hamiltonian form may include arbitrary generalized forces (or torques) — including dissipative or source terms.

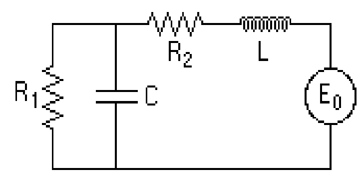
The most general form:

$$dp/dt = -\partial H(p,q)/\partial q + e(p,q,t)$$

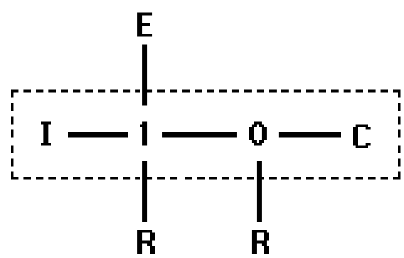
$$dq/dt = \partial H(p,q)/\partial p - f(p,q,t)$$

Hamilton's formulation applies to other energy domains

EXAMPLE: SIMPLE ELECTRIC CIRCUIT.



The energy storage elements may be distinguished from the rest of the system:



Generalized coordinates:

charge, q, and flux linkage, λ

Hamiltonian:

$$H(q,\lambda) = q^2/2C + \lambda^2/2L$$

Again, the Hamiltonian is the total system energy.

Gradients:

$$\partial H/\partial q = q/C$$

$$\partial H/\partial \lambda = \lambda/L$$

Without the source or resistors, the conservative Hamiltonian equations would be

$$e_L = d\lambda/dt = -\partial H/\partial q = -q/C$$

$$i_C = dq/dt = \partial H/\partial \lambda = \lambda/L$$

With the source and resistors, the *non-conservative* Hamiltonian equations are

$$d\lambda/dt = -\partial H/\partial q + E_0 - e_{R2}$$

$$dq/dt = \partial H/\partial \lambda - i_{R1}$$

CANONICAL TRANSFORMATIONS

Choice of (state) variables is important in physical system modeling

— particularly important for nonlinear mechanical systems

Geometry is fundamental. The choice of coordinates used to represent the geometry and kinematics of a system has a profound effect on the structure and complexity of its describing equations.

Transformations of state variables are used extensively to analyze linear state determined systems.

e.g., physical variables to diagonal form

Structure of the state equations is preserved while mathematical convenience is gained

e.g., in diagonalized form each equation is decoupled from the rest

— facilitates analysis, e.g. modal analysis

— proportionality between the rate and state vectors is preserved

DOES THIS APPLY TO THE NONLINEAR CASE?

The Hamiltonian form permits an analogous approach using *canonical transformations*.

Any change of variables that

— preserves the value of the Hamiltonian

— preserves the structure of the Hamilton's equations

is a canonical transformation.

EXAMPLE:

The "old" displacements may be functions of both the "new" displacements and/or the "new" momenta.

A simple canonical transformation

$p = -q^*$

$q = p^*$

old equations (conservative terms only)

$dp/dt = -\partial H(p,q)/\partial q$

$dq/dt = \partial H(p,q)/\partial p$

define $H^*(p^*,q^*) = H(p,q)$

— the value of the Hamiltonian is preserved

$\partial H^*/\partial p^* = (\partial H/\partial q)\,(\partial q/\partial p^*) = \partial H/\partial q$

$\partial H^*/\partial q^* = (\partial H/\partial p)\,(\partial p/\partial q^*) = -\,\partial H/\partial p$

time differentiate the new coordinates

$dq^*/dt = -\,dp/dt = \partial H/\partial q = \partial H^*/\partial p^*$

$dp^*/dt = dq/dt = \partial H(p,q)/\partial p = -\,\partial H^*/\partial q^*$

new equations (conservative terms only)

$dq^*/dt = \partial H^*/\partial p^*$

$dp^*/dt = -\,\partial H^*/\partial q^*$

— the structure of Hamilton's equations is preserved

new equations with non-conservative terms

$$dp^*/dt = -\partial H(p^*,q^*)/\partial q^* + e^*(p^*,q^*,t)$$

$$dq^*/dt = \partial H(p^*,q^*)/\partial p^* - f^*(p^*,q^*,t)$$

The forcing terms must be transformed as follows

$$f_k^* = \sum_j (e_j \partial p_j/\partial p_k^* + e_j \partial q_j/\partial p_k^*)$$

$$e_k^* = \sum_j (f_j \partial p_j/\partial q_k^* + e_j \partial q_j/\partial q_k^*)$$

NOTE:

Displacement and momentum may be exchanged

canonical transformation may destroy the physical meaning of variables

Parallel to transformation of a linear system to decoupled form

the original real-valued physical variables become complex valued