Chapter 5
Differential Motion

In the previous chapter, the position and orientation of the manipulator end-effecter were evaluated in relation to joint displacements. The joint displacements corresponding to a given end-effecter location were obtained by solving the kinematic equation for the manipulator. This preliminary analysis permitted the robotic system to place the end-effecter at a specified location in space. In this chapter, we are concerned not only with the final location of the end-effecter, but also with the *velocity* at which the end-effecter moves. In order to move the end-effecter in a specified direction at a specified speed, it is necessary to *coordinate* the motion of the individual joints. The focus of this chapter is the development of fundamental methods for achieving such coordinated motion in multiple-joint robotic systems. As discussed in the previous chapter, the end-effecter position and orientation are directly related to the joint displacements. Hence, in order to coordinate joint motions, we derive the *differential* relationship between the joint displacements and the end-effecter location, and then solve for the individual joint motions.

5.1 Differential Relationship

We begin by considering a two degree-of-freedom planar robot arm, as shown in Figure 5.1.1. The kinematic equations relating the end-effecter coordinates x_e and y_e to the joint displacements θ_1 and θ_2 are given by

$$x_e(\theta_1, \theta_2) = \ell_1 \cos\theta_1 + \ell_2 \cos(\theta_1 + \theta_2) \tag{5.1.1}$$

$$y_e(\theta_1, \theta_2) = \ell_1 \sin\theta_1 + \ell_2 \sin(\theta_1 + \theta_2) \tag{5.1.2}$$

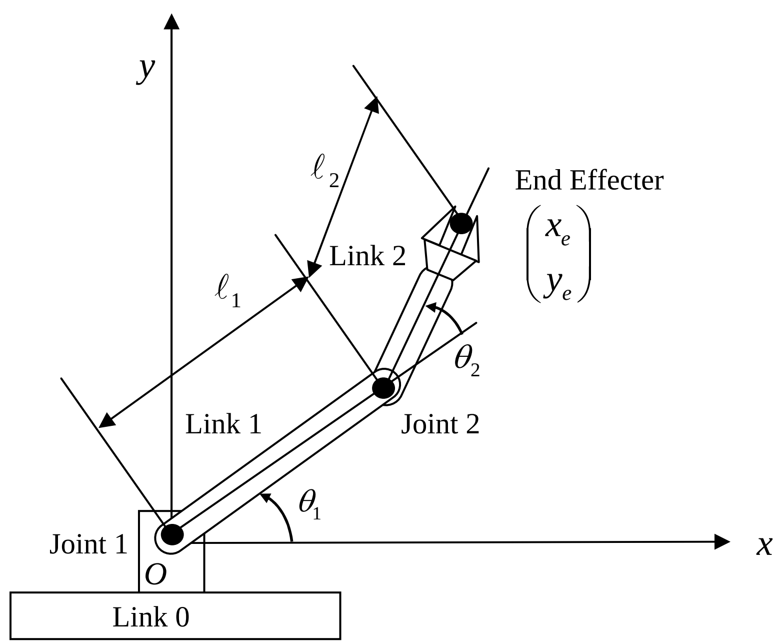

Figure 5.1.1 Two dof planar robot with two revolute joints

We are concerned with "small movements" of the individual joints at the current position, and we want to know the resultant motion of the end-effecter. This can be obtained by the total derivatives of the above kinematic equations:

$$dx_e = \frac{\partial x_e(\theta_1,\theta_2)}{\partial \theta_1} d\theta_1 + \frac{\partial x_e(\theta_1,\theta_2)}{\partial \theta_2} d\theta_2 \tag{5.1.3}$$

$$dy_e = \frac{\partial y_e(\theta_1,\theta_2)}{\partial \theta_1} d\theta_1 + \frac{\partial y_e(\theta_1,\theta_2)}{\partial \theta_2} d\theta_2 \tag{5.1.4}$$

where x_e, y_e are variables of both θ_1 and θ_2, hence two partial derivatives are involved in the total derivatives. In vector form the above equations reduce to

$$d\mathbf{x} = \mathbf{J} \cdot d\mathbf{q} \tag{5.1.5}$$

where

$$d\mathbf{x} = \begin{pmatrix} dx_e \\ dy_e \end{pmatrix}, \quad d\mathbf{q} = \begin{pmatrix} d\theta_1 \\ d\theta_2 \end{pmatrix} \tag{5.1.6}$$

and **J** is a 2 by 2 matrix given by

$$\mathbf{J} = \begin{pmatrix} \dfrac{\partial x_e(\theta_1,\theta_2)}{\partial \theta_1} & \dfrac{\partial x_e(\theta_1,\theta_2)}{\partial \theta_2} \\ \dfrac{\partial y_e(\theta_1,\theta_2)}{\partial \theta_1} & \dfrac{\partial y_e(\theta_1,\theta_2)}{\partial \theta_2} \end{pmatrix} \tag{5.1.7}$$

The matrix **J** comprises the partial derivatives of the functions $x_e(\theta_1,\theta_2)$ and $y_e(\theta_1,\theta_2)$ with respect to joint displacements θ_1 and θ_2. The matrix **J**, called the ***Jacobian Matrix,*** represents the differential relationship between the joint displacements and the resulting end-effecter motion. Note that most robot mechanisms have a multitude of active joints, hence a matrix is needed for describing the mapping of the vectorial joint motion to the vectorial end-effecter motion.

For the two-dof robot arm of Figure 5.1.1, the components of the Jacobian matrix are computed as

$$\mathbf{J} = \begin{pmatrix} -\ell_1 \sin\theta_1 - \ell_2 \sin(\theta_1+\theta_2) & -\ell_2 \sin(\theta_1+\theta_2) \\ \ell_1 \cos\theta_1 + \ell_2 \cos(\theta_1+\theta_2) & \ell_2 \cos(\theta_1+\theta_2) \end{pmatrix} \tag{5.1.8}$$

By definition, the Jacobian collectively represents the *sensitivities* of individual end-effecter coordinates to individual joint displacements. This sensitivity information is needed in order to coordinate the multi dof joint displacements for generating a desired motion at the end-effecter.

Consider the instant when the two joints of the robot arm are moving at joint velocities $\dot{\mathbf{q}} = (\dot{\theta}_1, \dot{\theta}_2)^T$, and let $\mathbf{v_e} = (\dot{x}_e, \dot{y}_e)^T$ be the resultant end-effecter velocity vector. The Jacobian provides the relationship between the joint velocities and the resultant end-effecter velocity. Indeed, dividing eq.(5) by the infinitesimal time increment *dt* yields

$$\frac{d\mathbf{x}_e}{dt} = \mathbf{J}\frac{d\mathbf{q}}{dt}, \quad \text{or} \quad \mathbf{v}_e = \mathbf{J} \cdot \dot{\mathbf{q}} \tag{5.1.9}$$

Thus the Jacobian determines the velocity relationship between the joints and the end-effecter.

5.2 Properties of the Jacobian

The Jacobian plays an important role in the analysis, design, and control of robotic systems. It will be used repeatedly in the following chapters. It is worth examining basic properties of the Jacobian, which will be used throughout this book.

We begin by dividing the 2-by-2 Jacobian of eq.(5.1.8) into two column vectors:

$$\mathbf{J} = (\mathbf{J}_1, \mathbf{J}_2), \quad \mathbf{J}_1, \mathbf{J}_2 \in \Re^{2\times 1} \tag{5.2.1}$$

Then eq.(5.1.9) can be written as

$$\mathbf{v}_e = \mathbf{J}_1 \cdot \dot{\theta}_1 + \mathbf{J}_2 \cdot \dot{\theta}_2 \tag{5.2.2}$$

The first term on the right-hand side accounts for the end-effecter velocity induced by the first joint only, while the second term represents the velocity resulting from the second joint motion only. The resultant end-effecter velocity is given by the vectorial sum of the two. Each column vector of the Jacobian matrix represents the end-effecter velocity generated by the corresponding joint moving at a unit velocity *when all other joints are immobilized*.

Figure 5.2.1 illustrates the column vectors $\mathbf{J}_1, \mathbf{J}_2$ of the 2 dof robot arm in the two-dimensional space. Vector $\mathbf{J}_2$, given by the second column of eq.(5.1. 8), points in the direction perpendicular to link 2. Note, however, that vector $\mathbf{J}_1$ is not perpendicular to link 1 but is perpendicular to line *OE*, the line from joint 1 to the endpoint *E*. This is because $\mathbf{J}_1$ represents the endpoint velocity induced by joint 1 when joint 2 is immobilized. In other words, links 1 and 2 are rigidly connected, becoming a single rigid body of link length *OE*, and $\mathbf{J}_1$ is the tip velocity of the link *OE*.

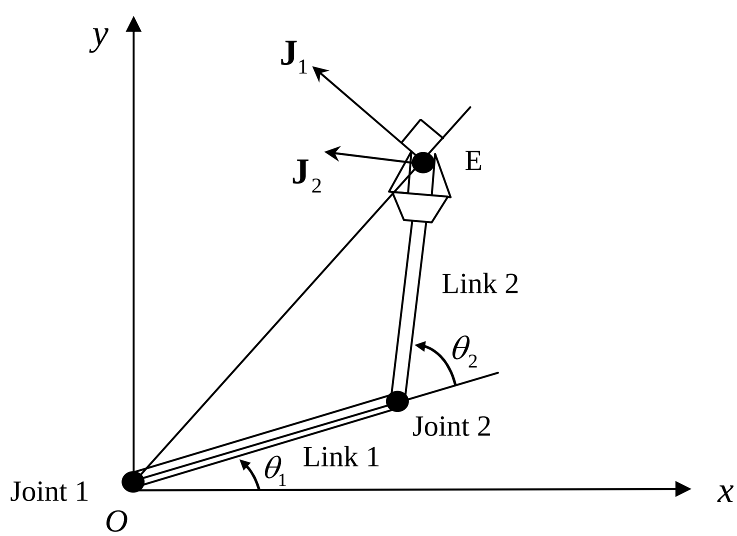

Figure 5.2.1 Geometric interpretation of the column vectors of the Jacobian

In general, each column vector of the Jacobian represents the end-effecter velocity and angular velocity generated by the individual joint velocity while all other joints are immobilized. Let $\dot{\mathbf{p}}$ be the end-effecter velocity and angular velocity, or the end-effecter velocity for short, and

$\mathbf{J}_i$ be the *i-th* column of the Jacobian. The end-effecter velocity is given by a linear combination of the Jacobian column vectors weighted by the individual joint velocities.

$$\dot{\mathbf{p}} = \mathbf{J}_1 \cdot \dot{q}_1 + \cdots + \mathbf{J}_n \cdot \dot{q}_n \tag{5.2.3}$$

where n is the number of active joints. The geometric interpretation of the column vectors is that $\mathbf{J}_i$ is the end-effecter velocity and angular velocity when all the joints other than joint i are immobilized and only the *i-th* joint is moving at a unit velocity.

Exercise Consider the two-dof articulated robot shown in Figure 5.2.1 again. This time we use "absolute" joint angles measured from the positive x-axis, as shown in Figure 5.2.2. Note that angle θ_2 is measured from the fixed frame, i.e. the x-axis, rather than a relative frame, e.g. link 1. Obtain the 2-by-2 Jacobian and illustrate the two column vectors on the xy plane. Discuss the result in comparison with the previous case shown in Figure 5.2.1.

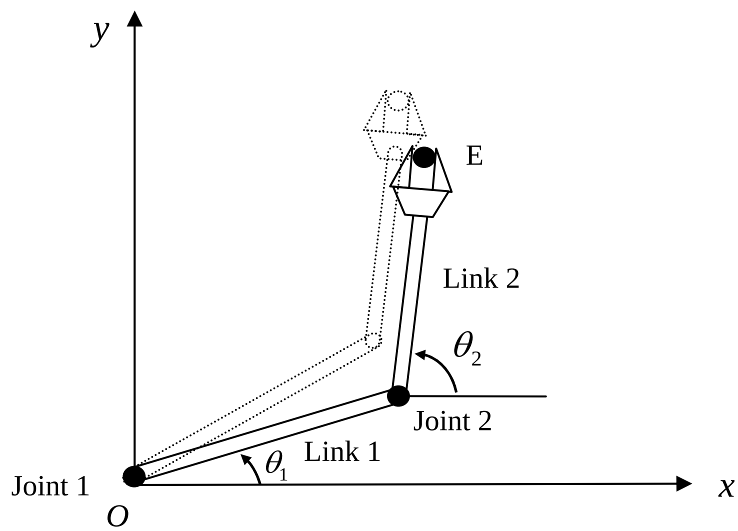

Figure 5.2.2 Absolute joint angles measured from the x-axis.

Note that the elements of the Jacobian are functions of joint displacements, and thereby vary with the arm configuration. As expressed in eq.(5.1.8), the partial derivatives, $\partial x_e / \partial \theta_i$, $\partial y_e / \partial \theta_i$, are functions of θ_1 and θ_2. Therefore, the column vectors $\mathbf{J}_1, \mathbf{J}_2$ vary depending on the arm posture. Remember that the end-effecter velocity is given by the linear combination of the Jacobian column vectors $\mathbf{J}_1, \mathbf{J}_2$. Therefore, the resultant end-effecter velocity varies depending on the direction and magnitude of the Jacobian column vectors $\mathbf{J}_1, \mathbf{J}_2$ spanning the two dimensional space. If the two vectors point in different directions, the whole two-dimensional space is covered with the linear combination of the two vectors. That is, the end-effecter can be moved in an arbitrary direction with an arbitrary velocity. If, on the other hand, the two Jacobian column vectors are aligned, the end-effecter cannot be moved in an arbitrary direction. As shown in Figure 5.2.3, this may happen for particular arm postures where the two links are fully contracted or extended. These arm configurations are referred to as singular configurations. Accordingly, the Jacobian matrix becomes ***singular*** at these positions. Using the determinant of a matrix, this condition is expressed as

$$\det \mathbf{J} = 0 \tag{5.2.4}$$

In fact, the Jacobian degenerates at the singular configurations, where joint 2 is 0 or 180 degrees. Substituting $\theta_2 = 0, \pi$ into eq.(5.1.8) yields

$$\det \mathbf{J} = \begin{pmatrix} -(\ell_1 \pm \ell_2)\sin\theta_1 & \mp \ell_2 \sin\theta_1 \\ (\ell_1 \pm \ell_2)\cos\theta_1 & \pm \ell_2 \cos\theta_1 \end{pmatrix} = 0 \tag{5.2.5}$$

Note that both column vectors point in the same direction and thereby the determinant becomes zero.

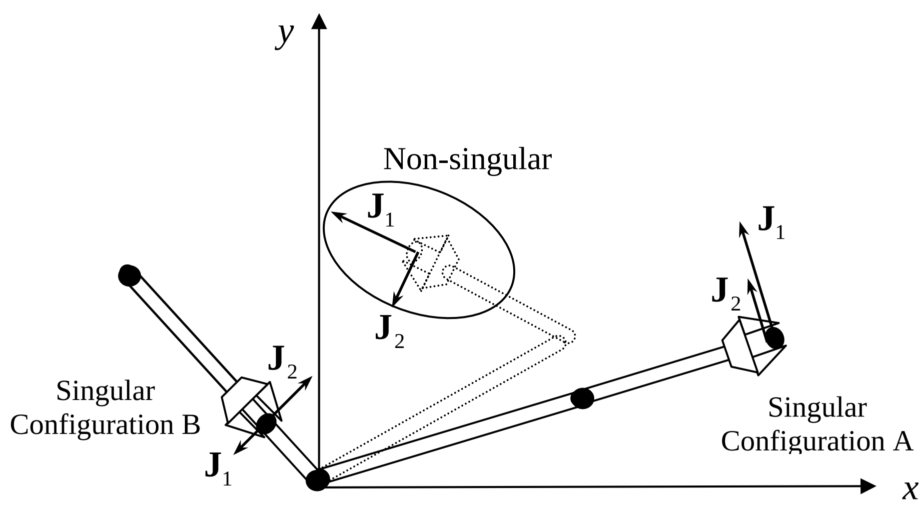

Figure 5.2.3 Singular configurations of the two-dof articulated robot

5.3 Inverse Kinematics of Differential Motion

Now that we know the basic properties of the Jacobian, we are ready to formulate the inverse kinematics problem for obtaining the joint velocities that allow the end-effecter to move at a given desired velocity. For the two dof articulated robot, the problem is to find the joint velocities $\dot{\mathbf{q}} = (\dot{\theta}_1, \dot{\theta}_2)^T$, for the given end-effecter velocity $\mathbf{v_e} = (v_x, v_y)^T$. If the arm configuration is not singular, this can be obtained by taking the inverse of the Jacobian matrix in eq.(5.1.9),

$$\dot{\mathbf{q}} = \mathbf{J}^{-1} \cdot \mathbf{v_e} \tag{5.3.1}$$

Note that the solution is unique. Unlike the inverse kinematics problem discussed in the previous chapter, the *differential* kinematics problem has a unique solution as long as the Jacobian is non-singular.

The above solution determines how the end-effecter velocity $\mathbf{v}_e$ must be decomposed, or *resolved*, to individual joint velocities. If the controls of the individual joints regulate the joint velocities so that they can track the resolved joint velocities $\dot{\mathbf{q}}$, the resultant end-effecter velocity will be the desired $\mathbf{v}_e$. This control scheme is called Resolved Motion Rate Control, attributed to Daniel Whitney (1969). Since the elements of the Jacobian matrix are functions of joint displacements, the inverse Jacobian varies depending on the arm configuration. This means that although the desired end-effecter velocity is constant, the joint velocities are not. Coordination is thus needed among the joint velocity control systems in order to generate a desired motion at the end-effecter.

Example Consider the two dof articulated robot arm again. We want to move the endpoint of the robot at a constant speed along a path staring at point *A* on the *x*-axis, (+2, 0), go around the origin through points *B* (+ε, 0) and *C* (0, +ε), and reach the final point *D* (0, +2) on the *y*-axis. See Figure 5.3.1. For simplicity, each arm link is of unit length. Obtain the profiles of the individual joint velocities as the end-effecter tracks the path at the constant speed.

Substituting $\mathbf{v_e} = (v_x, v_y)^T$ into eq.(1) yields

$$\dot{\theta}_1 = \frac{v_x \cos(\theta_1 + \theta_2) + v_y \sin(\theta_1 + \theta_2)}{\sin\theta_2} \tag{5.3.2}$$

$$\dot{\theta}_2 = \frac{v_x[\cos\theta_1 + \cos(\theta_1 + \theta_2)] + v_y[\sin\theta_1 + \sin(\theta_1 + \theta_2)]}{\sin\theta_2} \tag{5.3.3}$$

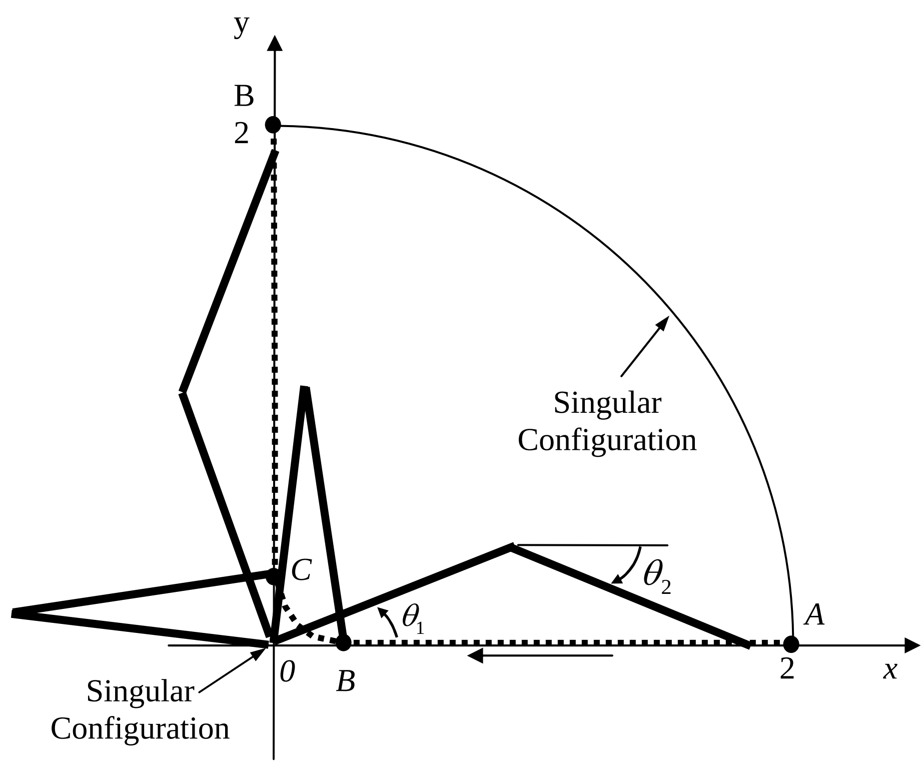

Figure 5.3.1 trajectory tracking near the singular points

Figure 5.3.2 shows the resolved joint velocities $\dot{\theta}_1, \dot{\theta}_2$ computed along the specified trajectory. Note that the joint velocities are extremely large near the initial and final points, and are unbounded at points A and D. These are at the arm's singular configurations, $\theta_2 = 0$. As the end-effecter gets close to the origin, the velocity of the first joint becomes very large in order to quickly turn the arm around from point B to C. At these configurations, the second joint is almost –180 degrees, meaning that the arm is near a singularity. This result agrees with the singularity condition using the determinant of the Jacobian:

$$\det \mathbf{J} = \sin\theta_2 = 0, \quad \therefore \theta_2 = k\pi, \quad k = 0, \pm 1, \pm 2, \cdots \tag{5.3.4}$$

In eqs.(2) and (3) above, the numerators are divided by $\sin\theta_2$, the determinant of the Jacobian. Therefore, the joint velocities $\dot{\theta}_1, \dot{\theta}_2$ blow up as the arm configuration gets close to the singular configuration.

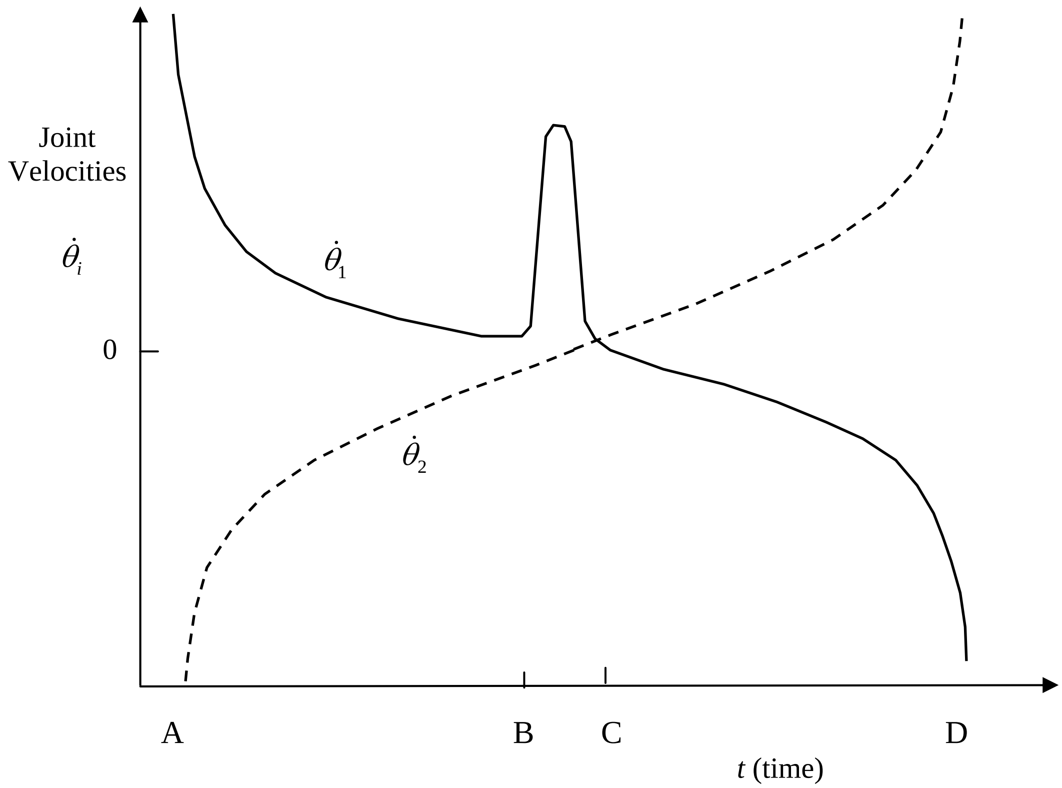

Figure 5.3.2 Joint velocity profiles for tracking the trajectory in Figure 5.3.1

Furthermore, the arm’s behavior near the singular points can be analyzed by substituting $\theta_2 = 0, \pi$ into the Jacobian, as obtained in eq.(5.2.5). For $\ell_1 = \ell_2 = 1$ and $\theta_2 = 0$, the Jacobian column vectors reduce to the ones in the same direction:

$$\mathbf{J}_1 = \begin{pmatrix} -2\sin\theta_1 \\ 2\cos\theta_1 \end{pmatrix}, \ \mathbf{J}_2 = \begin{pmatrix} -\sin\theta_1 \\ \cos\theta_1 \end{pmatrix}, \quad \mathit{for}\ \theta_2 = 0 \tag{5.3.5}$$

As illustrated in Figure 5.2.3 (singular configuration A), both joints $\dot{\theta}_1, \dot{\theta}_2$ generate the endpoint velocity along the same direction. Note that no endpoint velocity can be generated in the direction perpendicular to the aligned arm links. For $\theta_2 = \pi$,

$$\mathbf{J}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathbf{J}_2 = \begin{pmatrix} \sin\theta_1 \\ -\cos\theta_1 \end{pmatrix}, \quad \mathit{for}\ \theta_2 = \pi \tag{5.3.6}$$

The first joint cannot generate any endpoint velocity, since the arm is fully contracted. See singular configuration B in Figure 5.2.3.

At a singular configuration, there is at least one direction in which the robot cannot generate a non-zero velocity at the end-effecter. This agrees with the previous discussion; the Jacobian is degenerate at a singular configuration, and the linear combination of the Jacobian column vectors cannot span the whole space.

Exercise 5.2

A three-dof spatial robot arm is shown in the figure below. The robot has three revolute joints that allow the endpoint to move in the three dimensional space. However, this robot mechanism has singular points inside the workspace. Analyze the singularity, following the procedure below.

Step 1 Obtain each column vector of the Jacobian matrix by considering the endpoint velocity created by each of the joints while immobilizing the other joints.
Step 2 Construct the Jacobian by concatenating the column vectors, and set the determinant of the Jacobian to zero for singularity: $\det \mathbf{J} = 0$.
Step 3 Find the joint angles that make $\det \mathbf{J} = 0$.
Step 4 Show the arm posture that is singular. Show where in the workspace it becomes singular. For each singular configuration, also show in which direction the endpoint cannot have a non-zero velocity.

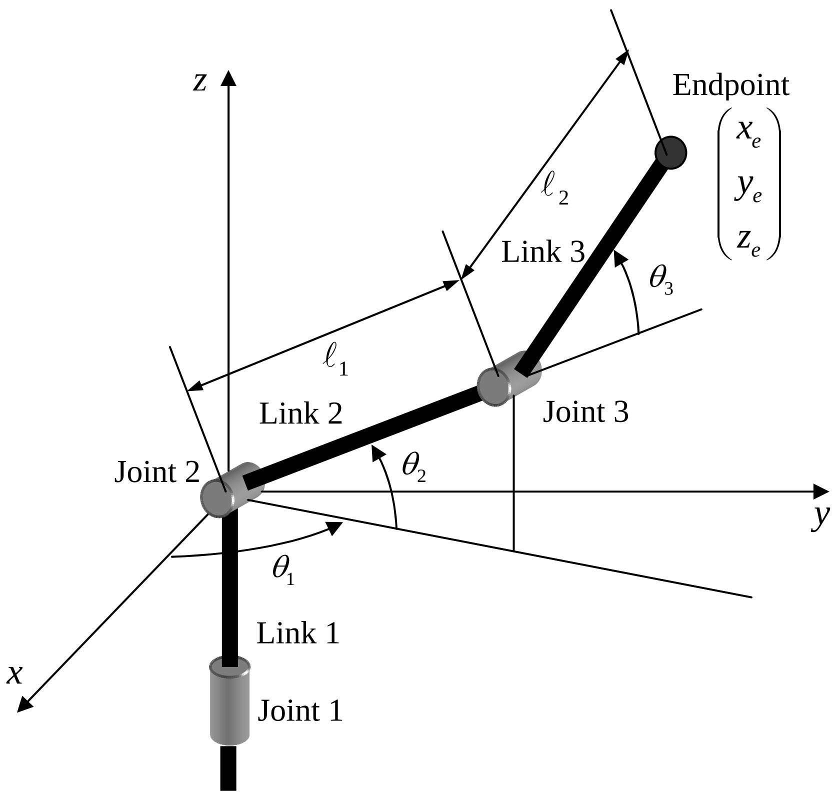

Figure 5.3.3 Schematic of a three dof articulated robot

5.4 Singularity and Redundancy

We have seen in this chapter that singular configurations exist for many robot mechanisms. Sometimes, such singular configurations exist in the middle of the workspace, seriously degrading mobility and dexterity of the robot. At a singular point, the robot cannot move in certain directions with a non-zero velocity. To overcome this difficulty, several methods can be considered. One is to plan a trajectory of the robot motion such that it will not go into singular configurations. Another method is to include additional degrees of freedom, so that even when some degrees of freedom are lost at a certain configuration, the robot can still maintain the necessary degrees of freedom. Such a robot is referred to as a redundant robot. In this section we will discuss singularity and redundancy, and obtain general properties of differential motion for general n degree of freedom robots.

As studied in Section 5.3, a unique solution exists to the differential kinematic equation, (5.3.1), if the arm configuration is non-singular. However, when a planar (spatial) robot arm has more than three (six) degrees of freedom, we can find an infinite number of solutions that provide the same motion at the end-effecter. Consider for instance the human arm, which has seven degrees of freedom excluding the joints at the fingers. When the hand is placed on a desk and fixed in its position and orientation, the elbow position can still vary continuously without moving the hand. This implies that a certain ratio of joint velocities exists that does not cause any velocity at the hand. This specific ratio of joint velocities does not contribute to the resultant endpoint velocity. Even if these joint velocities are superimposed onto other joint velocities, the resultant end-effecter velocity is the same. Consequently, we can find different solutions of the instantaneous kinematic equation for the same end-effecter velocity. In the following, we investigate the fundamental properties of the differential kinematics when additional degrees of freedom are involved.

To formulate a differential kinematic equation for a general n degree-of-freedom robot mechanism, we begin by modifying the definition of the vector $d\mathbf{x_e}$ representing the end-effecter motion. In eq. (5.1.6), $d\mathbf{x_e}$ was defined as a two-dimensional vector that represents the infinitesimal translation of an end-effecter. This must be extended to a general m-dimensional vector. For planar motion, m may be 3, and for spatial motion, m may be six. However, the number of variables that we need to specify for performing a task is not necessarily three or six. In arc welding, for example, only five independent variables of torch motion need be controlled. Since the welding torch is usually symmetric about its centerline, we can locate the torch with an arbitrary orientation about the centerline. Thus five degrees of freedom are sufficient to perform arc welding. In general, we describe the differential end-effecter motion by m independent variables $d\mathbf{p}$ that must be specified to perform a given task.

$$d\mathbf{p} = \begin{bmatrix} dp_1 & dp_2 & \cdots & dp_m \end{bmatrix}^T \in \Re^{m\times 1} \tag{5.4.1}$$

Then the differential kinematic equation for a general n degree-of-freedom robot is given by

$$d\mathbf{p} = \mathbf{J} \cdot d\mathbf{q} \tag{5.4.2}$$

where the dimension of the Jacobian $\mathbf{J}$ is m by n; $\mathbf{J} \in \Re^{m\times n}$. When n is larger than m and $\mathbf{J}$ is of full rank, there are *(n-m)* arbitrary variables in the general solution of eq.(2). The robot is then said to have *(n-m) redundant degrees of freedom* for the given task.

Associated with the above differential equation, the velocity relationship can be written as

$$\dot{\mathbf{p}} = \mathbf{J} \cdot \dot{\mathbf{q}} \tag{5.4.3}$$

where $\dot{\mathbf{p}}$ and $\dot{\mathbf{q}}$ are velocities of the end effecter and the joints, respectively.

Equation (3) can be regarded as a linear mapping from n-dimensional vector space $\boldsymbol{V^n}$ to m-dimensional space $\boldsymbol{V^m}$. To characterize the solution to eq.(3), let us interpret the equation using the linear mapping diagram shown in Figure 5.4.1. The subspace $R(\mathbf{J})$ in the figure is the range space of the linear mapping. The range space represents all the possible end-effecter velocities that can be generated by the *n* joints at the present arm configuration. If the rank of **J** is of full row rank, the range space covers the entire vector space $\boldsymbol{V^m}$. Otherwise, there exists at least one direction in which the end-effecter cannot be moved with non-zero velocity. The subspace $N(\mathbf{J})$ of Figure 5.4.1 is the null space of the linear mapping. Any element in this subspace is mapped into the zero vector in $\boldsymbol{V^m}$. Therefore, any joint velocity vector $\dot{\mathbf{q}}$ that belongs to the null space does not produce any velocity at the end-effecter. Recall the human arm discussed before. The elbow can move without moving the hand. Joint velocities for this motion are involved in the null space, since no end-effecter motion is induced. If the Jacobian is of full rank, the dimension of the null space, dim $N(\mathbf{J})$, is the same as the number of redundant degrees of freedom *(n-m)*. When the Jacobian matrix is degenerate, i.e. not of full rank, the dimension of the range space, dim $R(\mathbf{J})$, decreases, and at the same time the dimension of the null space increases by the same amount. The sum of the two is always equal to *n:*

$$\dim R(\mathbf{J}) + \dim N(\mathbf{R}) = n \tag{5.4.4}$$

Let $\dot{\mathbf{q}}^*$ be a particular solution of eq.(3) and $\dot{\mathbf{q}}_0$ be a vector involved in the null space, then the vector of the form $\dot{\mathbf{q}} = \dot{\mathbf{q}}^* + k\,\dot{\mathbf{q}}_0$ is also a solution of eq.(3), where *k* is an arbitrary scalar quantity. Namely,

$$\mathbf{J}\dot{\mathbf{q}} = \mathbf{J}\dot{\mathbf{q}}^* + k\mathbf{J}\dot{\mathbf{q}}_0 = \mathbf{J}\dot{\mathbf{q}}^* = \dot{\mathbf{p}} \tag{5.4.5}$$

Since the second term $k\dot{\mathbf{q}}_0$ can be chosen arbitrarily within the null space, an infinite number of solutions exist for the linear equation, unless the dimension of the null space is 0. The null space accounts for the arbitrariness of the solutions. The general solution to the linear equation involves the same number of arbitrary parameters as the dimension of the null space.

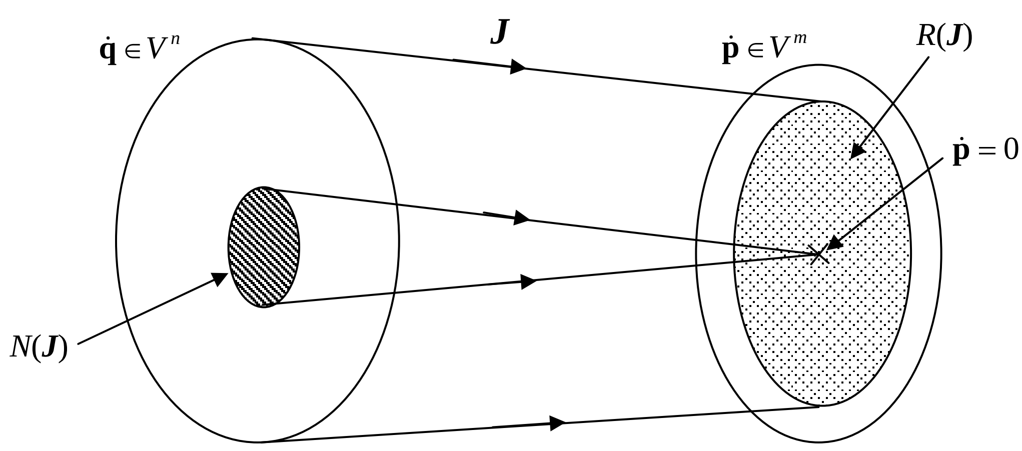

Figure 5.4.1 Linear mapping diagram