2.094

Finite Element Analysis of Solids and Fluids

Spring 2008

Quiz #2 - Solution

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Problem 1 (10 points)

a)

$$ {}_0^t\underline{X} = \begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix} \begin{bmatrix} 7/6 & 0 \\ 0 & 8/9 \end{bmatrix} = \begin{bmatrix} \frac{7\sqrt{3}}{12} & -\frac{4}{9} \\ \frac{7}{12} & \frac{4\sqrt{3}}{9} \end{bmatrix} $$

b)

$$ {}_0^t\underline{\varepsilon} = \frac{1}{2}\left({}_0^t\underline{X}^T \, {}_0^t\underline{X} - \underline{I} \right) = \frac{1}{2}\left({}_0^t\underline{U}^2 - \underline{I} \right) = \begin{bmatrix} \frac{13}{72} & 0 \\ 0 & -\frac{17}{162} \end{bmatrix} $$

$$ {}_0^t\underline{S} = E \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{pmatrix} \frac{13}{72} \\ -\frac{17}{162} \\ 0 \end{pmatrix} = E \begin{pmatrix} \frac{13}{72} \\ -\frac{17}{162} \\ 0 \end{pmatrix} $$

c)

$$ {}^t\underline{\tau} = \frac{{}^t\rho}{{}^0\rho} \, {}_0^t\underline{X} \, {}_0^t\underline{S} \, {}_0^t\underline{X}^T = \frac{1}{\det {}_0^t\underline{X}} \, {}_0^t\underline{X} \, {}_0^t\underline{S} \, {}_0^t\underline{X}^T = \frac{1}{\frac{7\sqrt{3}}{12}\cdot\frac{4\sqrt{3}}{9} + \frac{4}{9}\cdot\frac{7}{12}} \begin{bmatrix} \frac{7\sqrt{3}}{12} & -\frac{4}{9} \\ \frac{7}{12} & \frac{4\sqrt{3}}{9} \end{bmatrix} \begin{bmatrix} \frac{13}{72}E & 0 \\ 0 & -\frac{17}{162}E \end{bmatrix} \begin{bmatrix} \frac{7\sqrt{3}}{12} & \frac{7}{12} \\ -\frac{4}{9} & \frac{4\sqrt{3}}{9} \end{bmatrix} $$

Problem 2 (10 points)

The governing equation we need is

$$\int_V \bar{e}_{ij}\tau_{ij}dV = \Re$$

By substituting $\tau_{ij} = -p\delta_{ij} + \mu\left(v_{i,j} + v_{j,i}\right) = -p\delta_{ij} + 2\mu e_{ij}$,

$$\int_V \bar{e}_{ij}\tau_{ij}dV = \int_V \bar{e}_{ij}\left(2\mu\right)e_{ij}dV - \int_V \bar{e}_{ii}\,pdV$$

In the following calculations, we need to know the interpolation function and its derivatives only at the node 1.

$$h_1 = \frac{1}{2}x_1\left(1+x_1\right)x_2\left(1+2x_2\right)$$

$$h_{1,1} = \frac{1}{2}\left(1+2x_1\right)x_2\left(1+2x_2\right)$$

$$h_{1,2} = \frac{1}{2}x_1\left(1+x_1\right)\left(1+4x_2\right)$$

The volume integration of the element assuming the unit depth into the x_3 direction is,

$$\int_V (\quad)dV = \int_{-1}^{+1}\int_{-1/2}^{+1/2}(\quad)dx_1dx_2$$

a) Evaluate $K(1,1)$

Let

$$\underline{e} = \begin{pmatrix} e_{11} \\ e_{22} \\ 2e_{12} \end{pmatrix} = \begin{pmatrix} v_{1,1} \\ v_{2,2} \\ v_{1,2}+v_{2,1} \end{pmatrix} = \underline{B}\hat{\underline{v}} \quad \text{where } \hat{\underline{v}}^T = \begin{bmatrix} v_1^1 & v_2^1 & v_1^2 & v_2^2 & \cdots & v_1^9 & v_2^9 \end{bmatrix}$$

Then,

$$\int_V \bar{e}_{ij}\left(2\mu\right)e_{ij}dV = \hat{\bar{\underline{v}}}^T\left(\int_V \underline{B}^T\underline{C}\underline{B}dV\right)\hat{\underline{v}} \quad \text{where } \underline{C} = \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

We only need the first column of $\underline{B}$ to compute $K(1,1)$,

$$\underline{B}_{(1)}^T = \begin{bmatrix} h_{1,1} & 0 & h_{1,2} \end{bmatrix}$$

Therefore,

$$K(1,1) = \int_V \underline{B}^T_{(1)} \underline{C} \underline{B}_{(1)} dV = \int_V \left\{ 2\mu \left(h_{1,1} \right)^2 + \mu \left(h_{1,2} \right)^2 \right\} dV$$

*** Or, simply from Equation 7.79 in the textbook,

$$K_{\mu v_1 v_1} = \int_V \left(2\mu H^T_{,x_1} H_{,x_1} + \mu H^T_{,x_2} H_{,x_2} \right) dV$$

$$K(1,1) = K_{\mu v_1 v_1}(1,1) = \int_V \left\{ 2\mu \left(h_{1,1} \right)^2 + \mu \left(h_{1,2} \right)^2 \right\} dV$$

b) Evaluate $K(1,21)$

Let

$$\bar{e}_{ii} = \bar{e}_{11} + \bar{e}_{22} = \bar{v}_{1,1} + \bar{v}_{2,2} = \underline{B}_v \underline{\hat{v}}$$

$$p = p_0 + p_1 x_1 + p_2 x_2 = \begin{bmatrix} 1 & x_1 & x_2 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix} = \underline{\tilde{H}} \underline{\hat{p}}$$

Then,

$$-\int_V \bar{e}_{ii} p dV = \underline{\hat{\bar{v}}}^T \left(-\int_V \underline{B}^T_v \underline{\tilde{H}} dV \right) \underline{\hat{p}}$$

We need the first column of $\underline{B}_v$ and the third column of $\underline{\tilde{H}}$ to compute $K(1,21)$,

$$\underline{B}^T_{v(1)} = \left[h_{1,1} \right] \text{ and } \underline{\tilde{H}}_{(3)} = \left[x_2 \right]$$

Therefore,

$$K(1,21) = -\int_V \underline{B}^T_{v(1)} \underline{\tilde{H}}_{(3)} dV = -\int_V h_{1,1} x_2 dV$$

*** Or, simply from Equation 7.81 in the textbook,

$$K_{v_1 p} = -\int_V H^T_{,x_1} \tilde{H} dV$$

$$K(1,21) = K_{v_1 p}(1,3) = -\int_V h_{1,1} x_2 dV$$

c) The 9/3 element is a suitable element for the incompressible analysis because it does not lock and passes the inf-sup condition and therefore it shows an optimal convergence.

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