2.092/2.093 — Finite Element Analysis of Solids & Fluids I	Fall '09
Lecture 11 - Heat Transfer Analysis	
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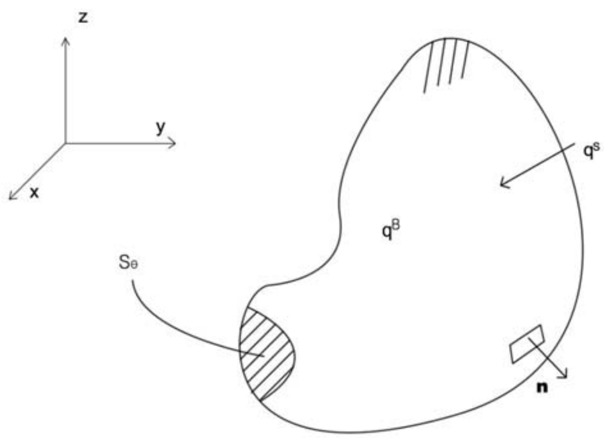
Reading assignment: Sections 7.1-7.4.1



To discuss heat transfer in systems, first let us define some variables.

$$\theta(x, y, z, t) = \text{Temperature}$$
$$S_\theta = \text{Surface area with prescribed temperature } (\theta_p)$$
$$S_q = \text{Surface area with prescribed heat flux into the body}$$

Given the geometry, boundary conditions, material laws, and loading, we would like to calculate the temperature distribution over the body. To obtain the exact solution of the mathematical model, we need to satisfy the following in the *differential formulation*:

- Heat flow equilibrium
- Compatibility
- Constitutive relation(s)

Example: One-Dimensional Case

We can derive an expression for system equilibrium from the heat flow equation.

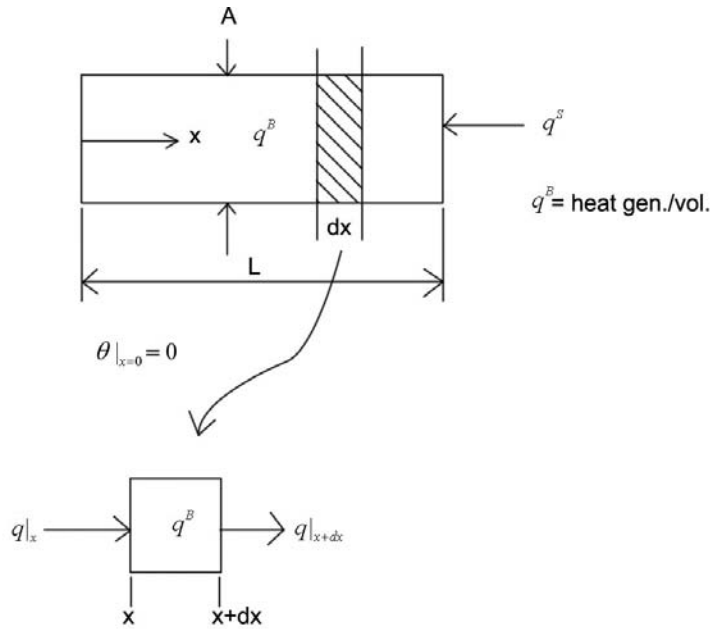
$$q\Big|_x - q\Big|_{x+dx} + q^B dx = 0$$

Using the constitutive equation,

$$q = -k\frac{\partial \theta}{\partial x}$$

$$q\Big|_{x+dx} = q\Big|_x + \frac{\partial q}{\partial x}\Big|_x dx$$

$$-k\frac{\partial \theta}{\partial x} + k\left(\frac{\partial \theta}{\partial x} + \frac{\partial^2 \theta}{\partial x^2}dx\right) + q^B dx = 0$$

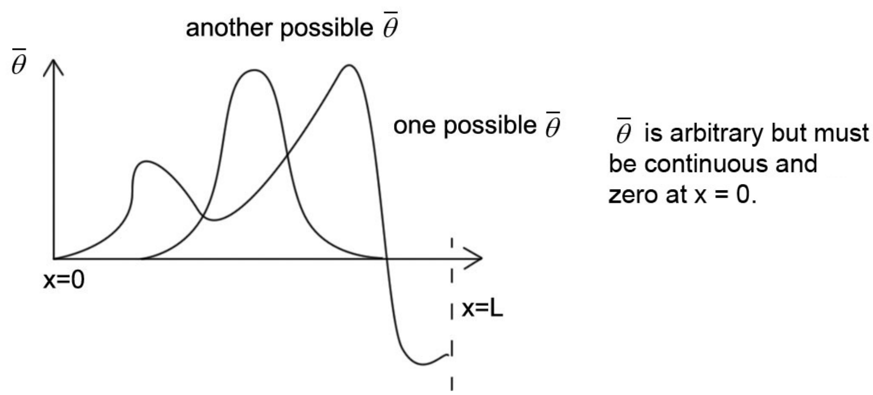


We obtain the result

$$k\frac{\partial^2\theta}{\partial x^2} + q^B = 0 \quad \text{in V}$$

$$k\frac{\partial\theta}{\partial x}\Big|_L = q^S\Big|_L \quad , \quad \theta\Big|_{x=0} = 0$$

Principle of Virtual Temperatures



Clearly:

$$\bar{\theta}\left(k\frac{\partial^2\theta}{\partial x^2} + q^B\right) = 0 \tag{A}$$

$$\int_V \bar{\theta}\left(k\frac{\partial^2\theta}{\partial x^2} + q^B\right) dV = A\int_0^L \bar{\theta}\left(k\frac{\partial^2\theta}{\partial x^2} + q^B\right) dx = 0$$

Hence,

$$\int_0^L \bar{\theta}\left(k\frac{\partial^2\theta}{\partial x^2} + q^B\right) dx = 0$$

$$\bar{\theta} k \frac{\partial \theta}{\partial x}\Big|_0^L - \int_0^L \left(\frac{\partial \bar{\theta}}{\partial x} k \frac{\partial \theta}{\partial x}\right) dx + \int_0^L \bar{\theta} q^B dx = 0 \tag{B}$$

$$\int_0^L \left(\frac{\partial \bar{\theta}}{\partial x}\right) k \left(\frac{\partial \theta}{\partial x}\right) dx = \int_0^L \bar{\theta} q^B dx + \left(\bar{\theta} q^S\right)\Big|_{x=L}$$

In 3D, the equation becomes

$$\int_V \bar{\boldsymbol{\theta}}'^T \boldsymbol{k} \boldsymbol{\theta}' dV = \int_V \bar{\theta}^T q^B dV + \int_{S_q} \bar{\theta}^T q^S dS_q \tag{C}$$

$$\boldsymbol{k} = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix} \quad ; \quad \boldsymbol{\theta}' = \begin{bmatrix} \frac{\partial \theta}{\partial x} \\ \frac{\partial \theta}{\partial y} \\ \frac{\partial \theta}{\partial z} \end{bmatrix} \quad ; \quad \bar{\boldsymbol{\theta}}' = \begin{bmatrix} \frac{\partial \bar{\theta}}{\partial x} \\ \frac{\partial \bar{\theta}}{\partial y} \\ \frac{\partial \bar{\theta}}{\partial z} \end{bmatrix}$$

For example:

$$q^S = h\left(\theta^e - \theta^S\right) \rightarrow \text{convection}$$

$$q^S = \kappa\left(\theta^r - \theta^S\right) \rightarrow \text{radiation}$$

$$\theta^e = \text{ temperature of the environment}$$
$$\theta^r = \text{ temperature of the radiation source}$$

$$\theta^{(m)}(x,y,z,t) = \boldsymbol{H}^{(m)} \boldsymbol{\theta} \quad ; \quad \theta^{S(m)} = \boldsymbol{H}^{S(m)} \boldsymbol{\theta}$$

$$\boldsymbol{\theta}'^{(m)} = \boldsymbol{B}^{(m)} \boldsymbol{\theta} \quad ; \quad \bar{\boldsymbol{\theta}}'^{(m)} = \boldsymbol{B}^{(m)} \bar{\boldsymbol{\theta}}$$

Substituting this into (C), we obtain

$$\boldsymbol{K}\boldsymbol{\theta} = \boldsymbol{Q}$$

$$\boldsymbol{K} = \sum_m \boldsymbol{K}^{(m)} \quad ; \quad \boldsymbol{K}^{(m)} = \int_{V^{(m)}} \boldsymbol{B}^{(m)T} \boldsymbol{k}^{(m)} \boldsymbol{B}^{(m)} dV^{(m)}$$

$$\boldsymbol{Q} = \boldsymbol{Q}_B + \boldsymbol{Q}_S$$

$$\boldsymbol{Q}_B = \sum_m \boldsymbol{Q}_B^{(m)} \quad ; \quad \boldsymbol{Q}_B^{(m)} = \int_{V^{(m)}} \boldsymbol{H}^{(m)T} \boldsymbol{q}^{B(m)} dV^{(m)}$$

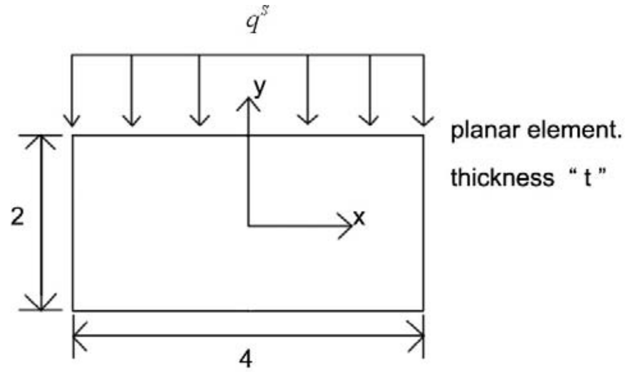
$$\boldsymbol{Q}_S = \sum_m \boldsymbol{Q}_S^{(m)} \quad ; \quad \boldsymbol{Q}_S^{(m)} = \int_{S_q^{(m)}} \boldsymbol{H}^{S^{(m)}T} \left(h\left(\theta^e - \theta^S\right)\right)^{(m)} dS_q{}^{(m)}$$

θ^S is unknown, so

$$\boxed{\boldsymbol{Q}_S^{(m)} = \int_{S_q^{(m)}} \boldsymbol{H}^{S^{(m)}T} h \theta^{e(m)} dS_q^{(m)} - \left[\int_{S_q^{(m)}} \boldsymbol{H}^{S^{(m)}T} h \boldsymbol{H}^{S^{(m)}} dS_q^{(m)}\right] \boldsymbol{\theta}}$$

Here we need to sum over all $S_q^{(m)}$ for element (m).

Example



$$\theta(x,y) = \begin{bmatrix} h_1 & h_2 & h_3 & h_4 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}$$

$$h_1 = \frac{1}{4}\left(1+\frac{x}{2}\right)(1+y) \quad ; \quad h_2 = \dots$$

$$\begin{bmatrix} \frac{\partial\theta}{\partial x} \\ \frac{\partial\theta}{\partial y} \end{bmatrix} = \underbrace{\begin{bmatrix} h_{1,x} & h_{2,x} & h_{3,x} & h_{4,x} \\ h_{1,y} & h_{2,y} & h_{3,y} & h_{4,y} \end{bmatrix}}_{\boldsymbol{B}} \boldsymbol{\theta}$$

$$\boldsymbol{H}^S = \boldsymbol{H}\,\big|_{y=1} \quad \rightarrow \quad \boldsymbol{H}^S = \frac{1}{2}\begin{bmatrix} \left(1+\frac{x}{2}\right) & \left(1-\frac{x}{2}\right) & 0 & 0 \end{bmatrix}$$