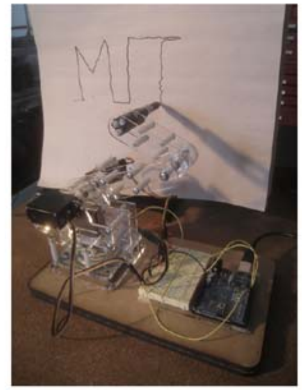
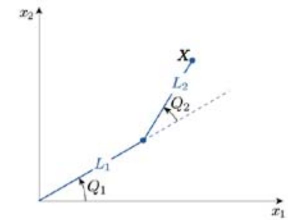
the Robot Arm

Geometry



actual

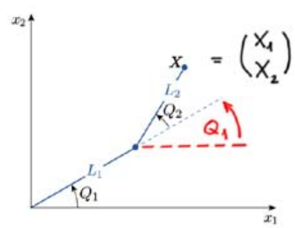


schematic

Q - X Relationship

$Q = (Q_1 \ Q_2)^T$ joint angles

$X = (X_1 \ X_2)^T$ end effector position (in plane)



$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} L_1\cos(Q_1) + L_2\cos(Q_1+Q_2) \\ L_1\sin(Q_1) + L_2\sin(Q_1+Q_2) \end{pmatrix}$$

$f^{\text{robot}}(q;X)$

$q = (q_1 \ q_2)^T$ any joint angles $0 \le q_1 < 2\pi$, $0 \le q_2 < 2\pi$

$X = (X_1 \ X_2)^T$ end effector position

$$f^{\text{robot}}(\underbrace{q}_{(q_1\ q_2)^T};\underbrace{X}_{(X_1\ X_2)^T}) \equiv \begin{pmatrix} f_1^{\text{robot}} \\ f_2^{\text{robot}} \end{pmatrix} = \begin{pmatrix} \overbrace{L_1\cos(q_1) + L_2\cos(q_1+q_2) - X_1}^{f_1^{\text{robot}}(q;X)} \\ \underbrace{L_1\sin(q_1) + L_2\sin(q_1+q_2) - X_2}_{f_2^{\text{robot}}(q;X)} \end{pmatrix}$$

$$\boxed{f^{\text{robot}}(Q;X) = 0 \iff Q\text{-}X \text{ relationship}}$$

Forward Kinematics^0: Q given $0 \le Q_1 < 2\pi,\ 0 \le Q_2 < 2\pi$

Given joint angles Q, find X_Q:

$$f^{robot}(Q; X_Q) = 0$$

$$\Downarrow$$

$$f_1^{robot}(Q; X_Q) = 0 \Rightarrow (X_Q)_1 = L_1 \cos Q_1 + L_2 \cos(Q_1 + Q_2)$$

$$f_2^{robot}(Q; X_Q) = 0 \Rightarrow (X_Q)_2 = L_1 \sin Q_1 + L_2 \sin(Q_1 + Q_2)$$

or

$$\boxed{X_Q = f^{robot}(Q; (0\ 0)^T)}\ .$$

linear and "explicit" (diagonal)

Inverse Kinematics^0: X given

Given end effector position X, find Q_X:

$$\boxed{f^{robot}(Q_X; X) = 0\ ; \quad \text{NONLINEAR in } Q_X}$$

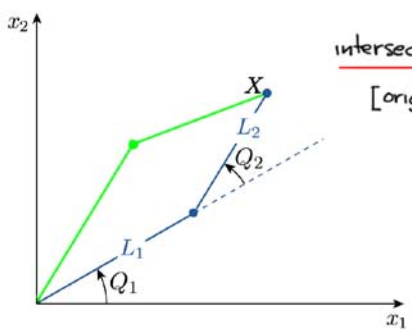
and

$$0 \le (Q_X)_1 < 2\pi\ ; \quad 0 \le (Q_X)_2 < 2\pi\ .$$

("normalization")

Multiple Solutions

often two solutions:



intersection of two circles:

$[\text{origin}; L_1] \cap [X; L_2]$

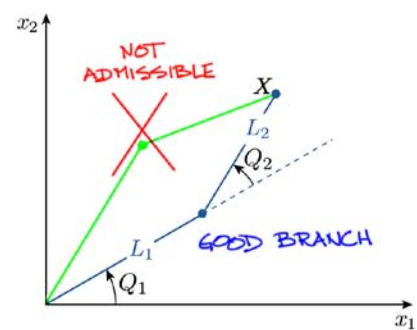
Sometimes one solution: e.g., $X = (L_1 + L_2)(\cos\theta\ \sin\theta)^T$

easily no solutions: e.g., $X = (L_1 + L_2 + 1)(\cos\theta\ \sin\theta)^T$

"Branch" Selection: Constraints on Solution

Actuator limits require

$$0 \le (Q_X)_1 \le \pi\ , \quad 0 \le (Q_X)_2 \le \pi\ .$$



Inverse Kinematics : X given

Given end effector position X, find Q_X:

$$\boxed{f^{\text{robot}}(Q_X; X) = 0; \quad \text{NONLINEAR in } Q_X}$$

and

$$0 \le (Q_X)_1 < 2\pi; \quad 0 \le (Q_X)_2 < 2\pi;$$

("normalization")

Newton's Method

then

AND

$$0 \le (Q_X)_1 \le \pi; \quad 0 \le (Q_X)_2 \le \pi.$$

("constraint")

CHECK

Newton's Method : given X

INITIAL GUESS $\hat{Q}_X$

while ($\| f^{\text{robot}}(\hat{Q}_X; X) \| >$ tolerance, and
number of iterations $\le$ max permitted)

$$J^{\text{robot}}(\hat{Q}_X; X)\, \widehat{\delta Q}_X = -f^{\text{robot}}(\hat{Q}_X; X)$$

$$\hat{Q}_X = \hat{Q}_X + \widehat{\delta Q}_X$$

$$\left.\begin{aligned} (\hat{Q}_X)_1 &= \mathrm{mod}((\hat{Q}_X)_1, 2\pi) \\ (\hat{Q}_X)_2 &= \mathrm{mod}((\hat{Q}_X)_2, 2\pi) \end{aligned}\right\} \text{normalization}$$

UPDATE norm, counter

end

CHECK CONSTRAINT: $0 \le (\hat{Q}_X)_1 \le \pi$ AND $0 \le (\hat{Q}_X)_2 \le \pi$?

where

$$J^{\text{robot}}(q; X) \equiv \begin{pmatrix} \dfrac{\partial f_1^{\text{robot}}}{\partial q_1}(\boldsymbol{q}; \boldsymbol{X}) & \dfrac{\partial f_1^{\text{robot}}}{\partial q_2}(\boldsymbol{q}; \boldsymbol{X}) \\ \dfrac{\partial f_2^{\text{robot}}}{\partial q_1}(\boldsymbol{q}; \boldsymbol{X}) & \dfrac{\partial f_2^{\text{robot}}}{\partial q_2}(\boldsymbol{q}; \boldsymbol{X}) \end{pmatrix}$$

Jacobian

for (recall)

$$\boldsymbol{f}^{\text{robot}}(\boldsymbol{q}; \boldsymbol{X}) \equiv \begin{pmatrix} f_1^{\text{robot}} \\ f_2^{\text{robot}} \end{pmatrix} = \begin{pmatrix} L_1 \cos(q_1) + L_2 \cos(q_1 + q_2) - X_1 \\ L_1 \sin(q_1) + L_2 \sin(q_1 + q_2) - X_2 \end{pmatrix}.$$

Possible Exit "States"

end; $\hat{Q}_X$

1. Satisfy

$$\| f^{\text{robot}}(\hat{Q}_X; X) \| \le \text{tolerance},$$

AND $0 \le (\hat{Q}_X)_1 \le \pi$, $0 \le (\hat{Q}_X)_2 \le \pi$;
(and)

2. Satisfy

$$\| f^{\text{robot}}(\hat{Q}_X; X) \| \le \text{tolerance},$$

BUT NOT $0 \le (\hat{Q}_X)_1 \le \pi$, $0 \le (\hat{Q}_X)_2 \le \pi$;

3. DO NOT satisfy

$$\| f^{\text{robot}}(\hat{Q}_X; X) \| \le \text{tolerance}.$$

Algorithmic Enhancements

$\Rightarrow$ Exit Condition 1.
"start close"

a) "Multi-Start"

Offline: pre-compute (forward kinematics)

(Q_k, X_k), $1 \le k \le M$ discrete workspace

$$\left.\begin{array}{l} 0 \le (Q_k)_1 \le \pi, \quad 0 \le (Q_k)_2 \le \pi; \\ \left.\begin{array}{l} f^{robot}(Q_k; X_k) = 0 \\ \quad\Downarrow \\ X_k = f^{robot}(Q_k; (0\ 0)^T) \end{array}\right] . \end{array}\right]$$

Online: choose initial guess as given X

$$\left.\begin{array}{l} \hat{Q} = Q_{k^*} \\ k^*: \| X - X_{k^*} \| \le \| X - X_k \|, \text{ all } k \ne k^* \end{array}\right\} . \quad \text{min } \cancel{\text{sort}}$$

b) Continuation

$X^{path;1}, X^{path;2}, \ldots$

$Q^{path;1}_{X^{path;1}} \to \hat{Q}$ initial guess

c) Homotopy

$X^{path;1}, X^{path;2}, \ldots$

$Q^{path;1}_{X^{path;1}} \to \hat{Q}$ initial guess

$\uparrow$

$X^{path;\text{"}3/2\text{"}}$

an Optimization Approach

Observation

Define

$$F^{robot}(Q; X) \equiv \| f^{robot}(Q; X) \|^2 .$$

Then, for given X,

$$\underline{f^{robot}(Q_X; X) = 0} \Rightarrow \underline{Q_X \text{ minimizes } F^{robot}(Q; X)};$$

however

$$\underline{Q^*_X \text{ minimizes } F^{robot}(Q; X)} \not\Rightarrow \underline{f^{robot}(Q^*_X; X) = 0} .$$

Approach

MATLAB fsolve

Given X, and an initial guess Q_X^0,

find a Q_X^* which minimizes $F^{robot}(Q;X)$;

then

if $\boxed{F^{robot}(Q_X^*;X) \leq \text{tolerance, and} \quad 0 \leq (Q_X)_1 \leq \pi,\ 0 \leq (Q_X)_2 \leq \pi}$

set $Q_X = Q_X^*$.

Advantages

robustness: decrease criterion;

incorporation of constraints

- equality
- inequality; MATLAB fmincon

BUT must check $F^{robot}(Q_X^*;X) \leq$ tolerance.

Note also: (naive) "squaring" can

- increase sensitivity
- decrease convergence rate.

Other applications: nonlinear least squares,

design optimization,

...