Session 12

Finish the example

I_n $[e_1\ e_2\ e_3]$

$$-I_1 \ddot{\varphi} \sin\delta = 0$$

$$\frac{1}{2}\dot{\varphi}^2 (I_3 - I_1)\sin 2\delta = N_B(l_1 + e\tan\delta) - N_A(l_2 - e\tan\delta) + T_A e$$

$$I_3 \ddot{\varphi} \cos\delta = 0$$

$$\Rightarrow \ddot{\varphi} = 0 \Rightarrow \dot{\varphi} = \omega_2 - \text{Const.} \quad \Rightarrow \varphi(t) = \varphi_0 + \omega_0 (t - t_0)$$

$$\Rightarrow \frac{1}{2}(I_3 - I_1)\omega_0^2 \sin 2\delta = N_B(l_1 + e\tan\delta) - N_A(l_2 - e\tan\delta) + T_A e$$

Linear momentum Principle:
$$\begin{cases} -em\,\omega_0^2 = N_A + N_B \\ T_A = mg \end{cases}$$

$$\Rightarrow N_B = \frac{\frac{1}{8}m\left(R^2 - \frac{h^2}{3}\right)\omega_0^2 \sin 2\delta - em\left[g + \omega_0^2 (l_2 - e\tan\delta)\right]}{l_1 + l_2}$$

$$N_A = \frac{em\left[g - \omega_0^2 (l_1 + e\tan\delta)\right] - \frac{1}{8}m\left(R^2 - \frac{h^2}{3}\right)\omega_0^2 \sin 2\delta}{l_1 + l_2}$$

Final Example on Newtonian Mechanics: Gyroscopes

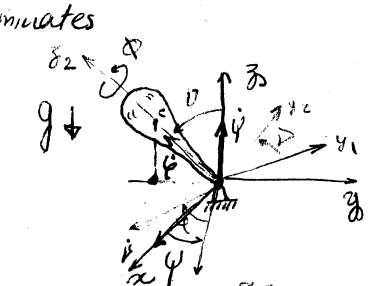
usual requirements in the definition of a gyroscope

(a) 3D rigid body with one of its point fixed

(b) In principal Coordinates, rotational symmetry is often assumed

$$\underline{\underline{I}}_c \cong \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{bmatrix}$$

(c) angular momentum about 3rd principal axes ($c\omega_3$) dominates



Euler axes & Euler angles use "3-1-3" Convention

"3": rotation about 3rd axis (z) by angle ψ (precession)

"1": rotation about 1st axis (x_1) by angle ν (nutation)

"3": rotation about 3rd axis (z_2) by angle φ (spin)

$$\omega = \underline{\dot{\psi}} + \underline{\dot{\nu}} + \underline{\dot{\phi}}$$

Angular Momentum Principle (about c)

$$\dot{\vec{H}}_c + \cancel{\underline{V}_c}^{\,0} \times \underline{P} = \underline{M}_c$$

→ torque of reaction force at O

Express $\underline{H}_c$ in principal Coordinate

$\vec{H}_c = \underline{\underline{I}}_c \, \underline{\omega}$; $\cancel{\vec{\omega} = \begin{bmatrix} \dot{\nu} \\ \dot{\psi} \sin \nu \\ \dot{\varphi} + \dot{\psi} \cos \nu \end{bmatrix}}$ $\quad \vec{\omega} = \begin{Bmatrix} \dot{\nu} \cos \varphi + \dot{\psi} \sin \nu \sin \varphi \\ \dot{\psi} \sin \nu \cos \varphi - \dot{\nu} \sin \varphi \\ \dot{\varphi} + \dot{\psi} \cos \nu \end{Bmatrix}$ ↙ Correct answer

$$= \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix}$$

$$\dot{\underline{H}}_c = \overset{\circ}{H}_c + \vec{\omega} \times \vec{H}_c$$

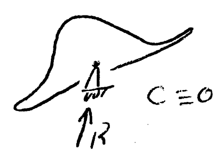
$$= \begin{Bmatrix} I_1 \dot{\omega}_1 \\ I_2 \dot{\omega}_2 \\ I_3 \dot{\omega}_3 \end{Bmatrix} + \begin{vmatrix} e_1 & e_2 & e_3 \\ \omega_1 & \omega_2 & \omega_3 \\ I_1 \omega_1 & I_2 \omega_2 & I_3 \omega_3 \end{vmatrix} \Rightarrow \boxed{\begin{cases} I_1 \dot{\omega}_1 + (I_3 - I_2)\omega_2 \omega_3 = M_1 \\ I_2 \dot{\omega}_2 + (I_1 - I_3)\omega_1 \omega_3 = M_2 \\ I_3 \dot{\omega}_3 + (I_2 - I_1)\omega_1 \omega_2 = M_3 \end{cases}}$$

M_i : expressed in principal Coordinate

Euler eq. for spinning top (euler's top)

Special Case $M_i = 0 \quad i = 1, 2, 3$

$$\begin{cases} I_1 \dot{\omega}_1 + (I_3 - I_2)\omega_2 \omega_3 = 0 \\ I_2 \dot{\omega}_2 + (I_1 - I_2)\omega_1 \omega_3 = 0 \\ I_3 \dot{\omega}_3 + (I_2 - I_1)\omega_1 \omega_2 = 0 \end{cases}$$



NOTE $\dot{\underline{H}}_c = \overset{\circ}{\underline{H}}_c = $ Const.

(2) $\dot{E} = \dot{T} + \cancel{\dot{V}}^{\,0} = 0$

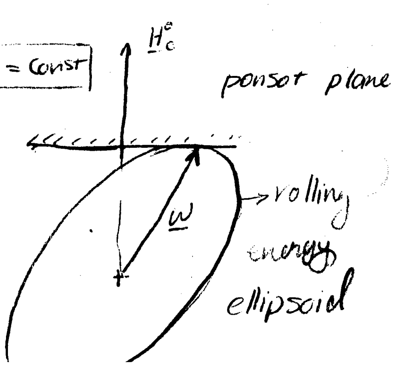
Reaction Force does no work $\quad E = E_0 = T_0 = $ Const

$$T = \frac{1}{2} m |V_c|^2 + \frac{1}{2} \underline{\omega}^T \overbrace{\underline{\underline{I}}_c \, \underline{\omega}}^{H_c} = E_0 = T_0 \Rightarrow \boxed{\underline{H}_c \cdot \underline{\omega} = 2T_0 = \text{Const}}$$

$$\Rightarrow I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2 = 2T_0$$

$$\frac{\omega_1^2}{\left(\frac{2T_0}{I_1}\right)} + \frac{\omega_2^2}{\left(\frac{2T_0}{I_2}\right)} + \frac{\omega_3^2}{\left(\frac{2T_0}{I_3}\right)} = 1 \longrightarrow \text{Energy Ellipsoid}$$

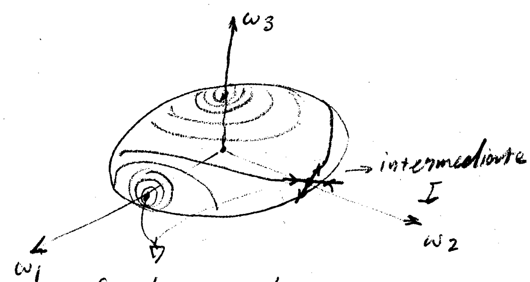
$\underline{\omega}$ must "ru"



$\Rightarrow$ trajectories (orbits) of euler's spinning top form curves on the energy ellipsoid

trajectories on ellipsoid are called "polliods"

$I_1 < I_2 < I_3$



fixed points of the eq. of motion

$\omega_1 = 0$	$\omega_1 =$ Conste	$\omega_1 = 0$
$\omega_2 =$ Cte	$\omega_2 = 0$	$\omega_2 = 0$
$\omega_3 = 0$	$\omega_3 = 0$	$\omega_3 =$ Const

Rotation about intermediate axis is unstable others are Stable

Fixed points (equilibria) for moment-free top

(1) $\omega_1 = \omega_2 = 0, \ \omega_3 \neq 0 \quad (\pm)$ (from energy Conservation we got two answers

(2) $\omega_1 = \omega_3 = 0 \quad \omega_2 \neq 0 \quad (\pm)$

(3) $\omega_2 = \omega_3 = 0 \quad \omega_1 = 0 \quad (\pm)$

Linearized eq. of motion

$$\begin{Bmatrix} \dot\omega_1 \\ \dot\omega_2 \\ \dot\omega_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} \cancel{\frac{I_2-I_3}{I_1}} & \frac{I_2-I_3}{I_1}\omega_{30} & \frac{I_2-I_3}{I_1}\omega_{20} \\ \frac{I_3-I_1}{I_2}\omega_{30} & 0 & \frac{I_3-I_1}{I_2}\omega_{10} \\ \frac{I_1-I_2}{I_3}\omega_{20} & \frac{I_1-I_2}{I_3}\omega_{10} & 0 \end{bmatrix}}_{\text{jacobian nonlinear terms at equilibria}} \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix}$$

(1) $\Rightarrow$ $$J = \begin{bmatrix} 0 & \frac{I_2-I_3}{I_1}\omega_{30} & 0 \\ \frac{I_3-I_1}{I_2}\omega_{30} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

eigen values: $\lambda_1 = 0$

$$\lambda_2, \lambda_3 = \pm\sqrt{\frac{(I_2-I_3)(I_3-I_1)}{I_1 I_2}}\,\omega_{30} = \pm i\alpha$$

Oscillations about ω_3 axis.

(2) $\Rightarrow$ $\lambda_1 = 0$

$$\lambda_{2,3} = \pm\sqrt{\frac{(I_2-I_3)(I_1-I_2)}{I_1 I_2}} = \pm\beta$$

Saddle type behavior about ω_2 axis

(3) Similar