2.004 MODELING DYNAMICS AND CONTROL II Spring 2003

Solutions to Problem Set No. 4

Problem 1 This problem can be solved by using the equation to get the velocity at any point in the 2D rigid body. The equation in this problem is

$$\overrightarrow{v_A} = \overrightarrow{v_A} + \vec{\omega} \times \vec{r}$$

Because the ladder slides down, the direction of the angular velocity is counter clockwise normal to the out of the face. Once we set the magnitude of the angular velocity as ω, the direction of the cross product of ω and r can be shown in the Figure 1. However, we don't know the magnitude of the cross product yet. Also, we know the direction of velocity of point A due to the geometric constraint between the point A and the vertical wall. It's the vertical direction as shown in the Figure 1.

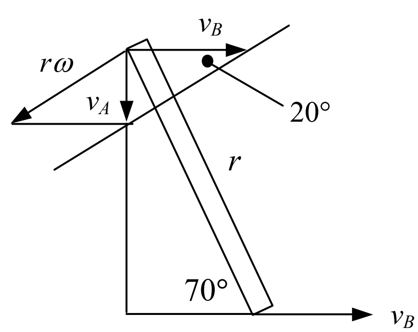

Figure 1

From the triangle includes v_A and v_B with angle 20 degree as shown in Figure 1,

$$v_A = v_B \tan(20°) = 2\tan(20°) = 0.73(m/s)$$

Problem 2. Motion of a particle confined to a pivoted circular tube. The circular tube, pivoted at the fixed point O, and constrained to remain in the plane of the sketch, has a single degree of freedom. The particle m, constrained to slide inside the tube, has an additional degree of freedom.

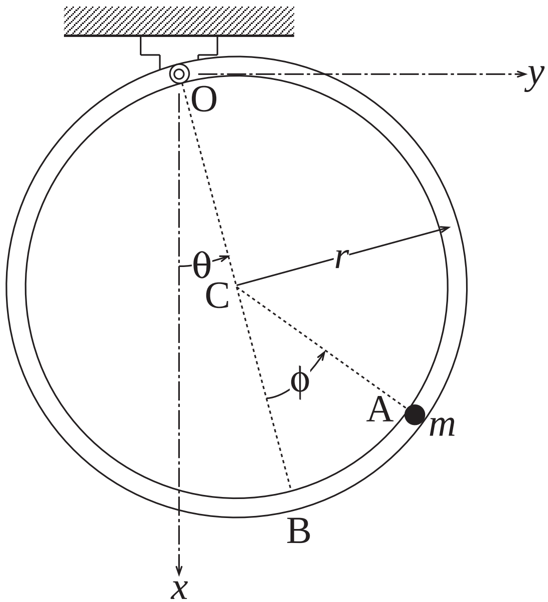

Figure 2: Pendulous circular tube carries mass particle m.

The location of the circular tube is fixed by giving the orientation of a stripe painted on the tube with respect to a reference direction. Here it is convenient to use the stripe OB which is the diameter of the tube which is vertical when the tube hangs down in its lowest position, and to use the vertical direction Ox as the reference direction. The position of the circular tube is then fixed by giving the angle θ. The position A of the particle m is then fixed by giving the angle ϕ. The position of the particle with respect to the fixed coordinate system Oxy is

$$x = r\cos\theta + r\cos(\theta+\phi) \qquad \text{and} \qquad y = r\sin\theta + r\sin(\theta+\phi)$$

The velocity components of the particle are obtained by differentiating these displacement components with respect to time.

$$\dot{x} = -r\dot{\theta}\sin\theta - r(\dot{\theta}+\dot{\phi})\sin(\theta+\phi) \qquad \text{and} \qquad \dot{y} = r\dot{\theta}\cos\theta + r(\dot{\theta}+\dot{\phi})\cos(\theta+\phi)$$

These components can be added vectorially to give the result sketched below.

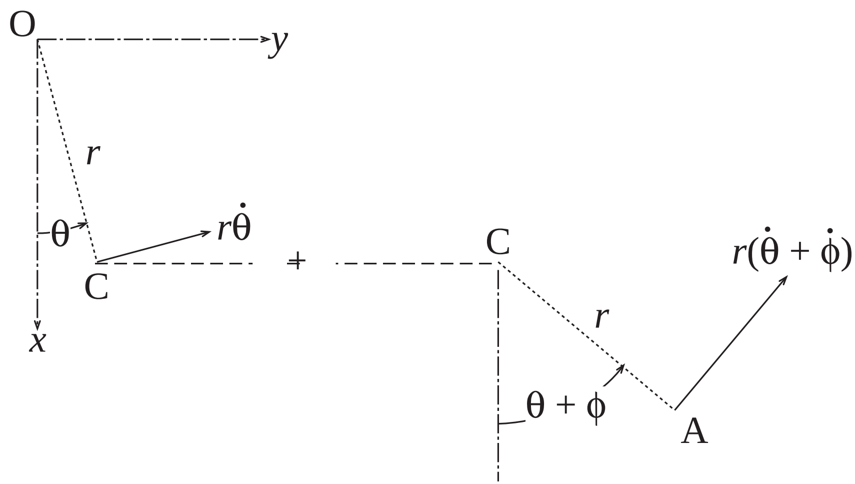

Figure 3 Vector summation of velocity components.

The total velocity of the mass particle at point A is the vector sum of two vectors. The first vector has the magnitude $r\dot{\theta}$ and is perpendicular to the line OC. An alternate procedure for obtaining this vector is to apply the general vector formula derived in class to the rigid element OC

$$\vec{v}_C = \vec{v}_O + \vec{\omega} \times \vec{r}$$

where here, $\vec{v}_O = 0$ and $\vec{\omega}$, the angular velocity of the rigid element OC, is $\dot{\theta}$ in the counter-clockwise direction, and $\vec{r}$ is the displacement vector $\overrightarrow{OC}$.

The second vector has the magnitde $r(\dot{\theta} + \dot{\phi})$ and is perpendicular to the rigid element CA. It represents the second term in the general formula

$$\vec{v}_A = \vec{v}_C + \omega \times \vec{r}$$

where, now, $\vec{\omega}$ is the angular velocity $\dot{\theta} + \dot{\phi}$ in the counter-clockwise direction of the rigid element CA, and $\vec{r}$ is the displacement vector $\overrightarrow{CA}$.

Problem 3 Velocities in crank and connecting-rod mechanism. In the given problem the lengths a and b were equal and the angle θ had the special values 0 and $\pi/2$. For these special cases the problem can be solved quite simply. In the general case, where a and b are different and θ is an arbitrary angle, the kinematical analysis of the crank and connecting-rod mechanism is surprisingly difficult. Here the general case is outlined and then a simple direct solution for the special cases is described. Although the system has only one degree of freedom θ, it is convenient to do the preliminary analysis in terms of the two angles θ and ϕ, and then later introduce the connection between ϕ and θ.

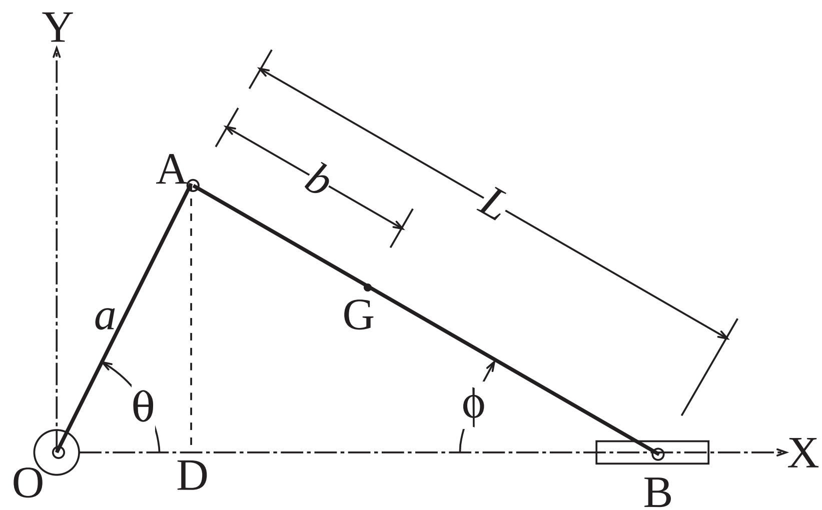

Figure 4 Crank and Connecting Rod: General Case.

The X- and Y-components of the displacement of the point G are

$$x_G = a\cos\theta + b\cos\phi \qquad \text{and} \qquad y_G = a\sin\theta - b\sin\phi$$

The components of the velocity of G are

$$\dot{x}_G = -a\dot{\theta}\sin\theta - b\dot{\phi}\sin\phi \qquad \text{and} \qquad \dot{y}_G = a\dot{\theta}\cos\theta - b\dot{\phi}\cos\phi$$

The connection between ϕ and θ is obtained by expressing the length AD in two ways

$$a\sin\theta = L\sin\phi \tag{1}$$

from which it follows that

$$\sin\phi = \frac{a}{L}\sin\theta \qquad \text{and} \qquad \cos\phi = \sqrt{1 - (\frac{a}{L}\sin\theta)^2}$$

The angular velocity of the crank OA is $\dot{\theta}$ counter-clockwise and the angular velocity of the connecting rod is $\dot{\phi}$ clockwise. The connection between $\dot{\phi}$ and $\dot{\theta}$ is obtained by differentiating Eq.(1).

$$a\dot{\theta}\cos\theta = L\dot{\phi}\cos\phi \qquad \text{or} \qquad \dot{\phi} = \frac{a\ cos\theta}{L\cos\phi}\dot{\theta}$$

The answers to parts (a), (b), (c), and (d) can be obtained by substituting

$$b = a, \qquad L = 2a, \qquad \text{and} \qquad \theta = 0, \quad \text{or} \quad \pi/2$$

into the above formulas. More directly, these special cases can be solved as follows.

(a) When $\theta = 0$ (so-called "top, dead center" position), the velocity of point A is $a\dot{\theta}$, vertically upward, and the velocity of point B is instantaneously zero. Under these conditions, the connecting rod AB is moving, at that instant, the same as an identical fictitious rod that rotates about a fixed axis through B with angular velocity $\omega_{AB} = a\dot{\theta}/2a$, clockwise. On that fictitious rod, the velocity of point G is clearly $\frac{a\dot{\theta}}{2}$, vertically upward. At this instant this is also the velocity of point G on the actual rod.

(b) When $\theta = 0$, with the velocity at A equal to $a\dot{\theta}$, and the velocity of B equal to zero, the angular velocity of the actual rod AB is the same as that of the fictitious rod described in (a).

$$\omega_{AB} = \frac{a\dot{\theta}}{2a} = \frac{\dot{\theta}}{2} \text{ clockwise}$$

(c) When B is not at dead center,the velocity of B is constrained to be horizontal. When $\theta = \pi/2$ the velocity of point A is $a\dot{\theta}$, horizontally toward the left. If A and B are both moving horizontally, the only way AB can remain rigid is for the entire rod to be translating with the velocity $a\dot{\theta}$ and not rotating. The velocity of G is $a\dot{\theta}$ horizontally to the left.

(d) The angular velocity of AB when $\theta = \pi/2$ is zero.

Problem 4 Motion of three-degee-of-freedom system. The variables x, ϕ, and ψ form a complete and independent set of generalized coordinates for the system shown in Fig.5.

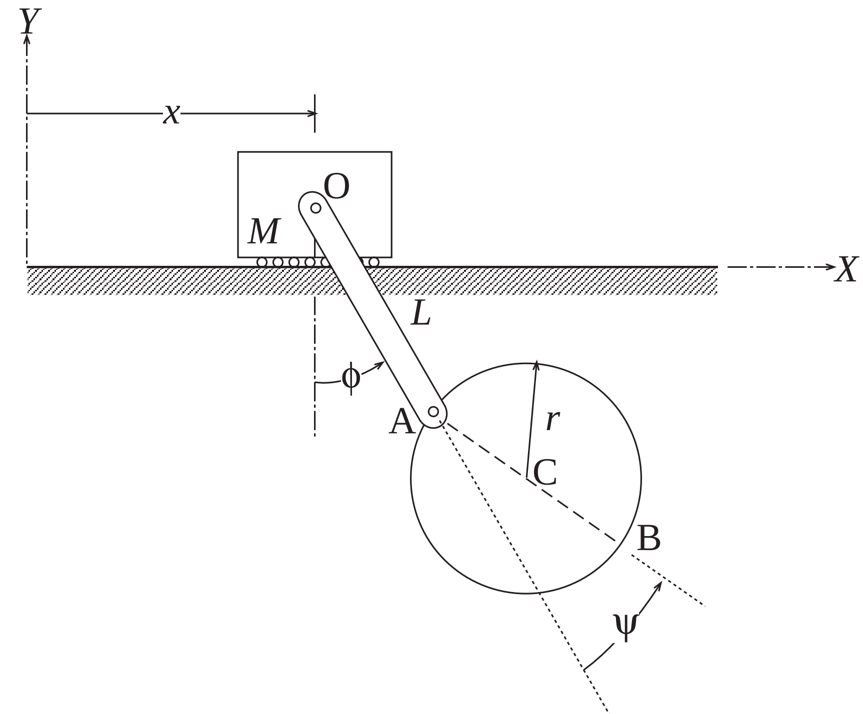

Figure 5: Three rigid bodies constrained to have three degrees of freedom.

(a) To find the angular velocity of the disk, paint a line along the diameter AB and observe its rate of change in orientation with respect to a fixed reference direction. Taking the reference direction to be vertical the angle made by AB is $\phi + \psi$ and thus

$$\omega_{disk} = \dot{\phi} + \dot{\psi}$$

(b) To find the velocity components of the point A, first obtain the displacement components and then differente them with respect to time. From Fig.5 it is seen that

$$X_A = x + L \sin \phi \qquad \text{and} \qquad Y_A = y_O - L \cos \phi$$

where $y_O$is the *constant* height of the pivot point O above the X-axis. The components of the velocity of A are

$$v_X = \dot{X}_A = \dot{x} + L\dot{\phi} \cos \phi \qquad \text{and} \qquad v_Y = \dot{Y}_A = L\dot{\phi} \sin \phi$$