2.004 MODELING DYNAMICS AND CONTROL II Spring 2003

Solutions to Problem Set No. 2

Problem 1. Impact of Vehicles with Shock Absorbers. In the first test a single vehicle traveling with velocity v_o is crashed into the barrier. The vehicle is modelled as a mass m with a shock absorber consisting of a spring and a damper in parallel. During the impact let the (compressive) deflection of the shock absorber be $x(t)$ and the velocity of the vehicle (positive in the direction of the initial velocity v_o) be $v(t)$. Then geometric compatibility requires

$$\frac{dx}{dt} = v \tag{1}$$

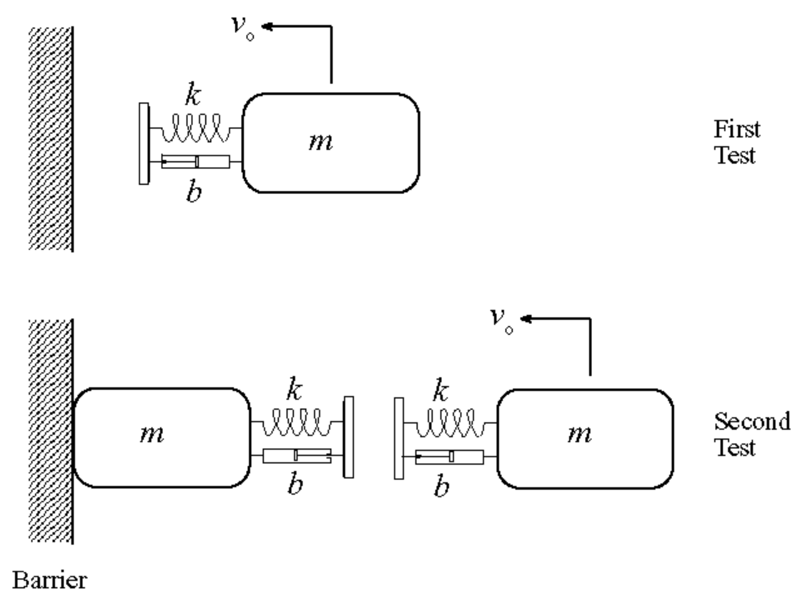

Figure 1: Crash Tests of Vehicles

The constitutive equations for the spring, dashpot, and mass are

$$f_k = kx \qquad \text{and} \qquad f_b = bv \qquad \text{and} \qquad f_m = m\frac{dv}{dt} \tag{2}$$

The force balance relation is

$$-f_k - f_b = f_m \qquad \text{or} \qquad f_m + f_b + f_k = 0 \tag{3}$$

When the constitutive equations (2) are inserted in force-balance relation (3) the dynamic equation becomes

$$m\frac{dv}{dt} + bv + kx = 0 \tag{4}$$

The two equations (1) and (4) are state equations for the two state variables x and v. They can be combined into a single matrix equation

$$\frac{d}{dt}\left\{\begin{array}{c} x \\ v \end{array}\right\} = \left[\begin{array}{cc} 0 & 1 \\ -k/m & -b/m \end{array}\right]\left\{\begin{array}{c} x \\ v \end{array}\right\} \tag{5}$$

The initial conditions for the motion during the impact are

$$x(0) = 0 \qquad \text{and} \qquad v(0) = v_o \tag{6}$$

(a) From Eq.(5) we recognize

$$\omega_o^2 = \frac{k}{m} \qquad \text{and} \qquad 2\zeta\omega_o = \frac{b}{m} \tag{7}$$

and then obtain

$$\zeta = \frac{b}{2\sqrt{mk}} \tag{8}$$

(b) When the initial velocity is raised to $v(0) = 2v_o$, the equation (5) remains unchanged so the relations (7) and (8) are unchanged.

(i) The damping ratio ζ is the same for both initial speeds.

(ii) The undamped natural frequency ω_o is the same for both initial speeds.

Since the equation (5) is *linear* the only effect of an increase in the initial velocity in Eq.(6) is to scale up the response magnitude in proportion to the input velocity. The period of damped vibration remains unchanged and the fraction of that period in which the shock absorber is in compression remains unchanged. The collision is over as soon as the model above requires the shock absorber to *pull* on the barrier.

(iii) The collision duration is the same for both initial speeds.

A formal analytical demonstration that the collision duration is independent of the initial velocity is obtained by reducing the equations to a dimensionless form. Intoduce a dimensionless velcity $V = v/v_o$ and a dimensionless displacement $X = x\omega_o/v_o$ into Eq.(5) to get

$$\frac{d}{dt}\left\{\begin{array}{c} X \\ V \end{array}\right\} = \left[\begin{array}{cc} 0 & \omega_o \\ -k/m\omega_o & -b/m \end{array}\right]\left\{\begin{array}{c} X \\ V \end{array}\right\} \tag{9}$$

The initial conditions for these state equations for the dimensionless variables X and V are

$$X(0) = 0 \qquad \text{and} \qquad V(0) = 1 \tag{10}$$

The solution to (9) subject to the initial conditions (10) is independent of the initial velocity v_o, yet from that solution the response for any particular initial velocity can be be inferred by the simple scaling operations

$$x(t) = \frac{v_o}{\omega_o}X(t) \qquad \text{and} \qquad v(t) = v_oV(t) \tag{11}$$

In the second test the moving vehicle crashes into an identical vehicle parked in contact with the barrier. During the impact the parked vehicle does not move but its shock absorber is connected in *series* with the shock absorber of the moving vehicle. This means that the same force is transmitted through both shock absorbers and that the total motion of the moving vehicle is the *sum* of the motions of the two shock absorbers. A spring which deflects twice the distance of a spring with spring constant k under the same force, has a spring constant of $k/2$. Similarly a damper that permits twice the velocity of a damper with damping constant b under the same force, has a damping constant of $b/2$. The state equations for the second test are similar to those for the first test. The only difference is that k is replaced by $k/2$, and b is replaced by $b/2$.

(c) Changing the stiffness and damping parameters in (5) in this manner yields

$$\frac{d}{dt}\begin{Bmatrix} x \\ v \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ k/2m & b/2m \end{bmatrix}\begin{Bmatrix} x \\ v \end{Bmatrix} \tag{12}$$

The initial conditions for the motion during the impact remain the same

$$x(0) = 0 \qquad \text{and} \qquad v(0) = v_o \tag{13}$$

(d) The behavioral parameters for the second test are obtained from

$$\omega_o^2 = \frac{k}{2m} \qquad \text{and} \qquad 2\zeta\omega_o = \frac{b}{2m} \tag{14}$$

The results are

$$\omega_o = \frac{1}{\sqrt{2}}\sqrt{\frac{k}{m}} \qquad \text{and} \qquad \zeta = \frac{1}{\sqrt{2}}\frac{b}{2\sqrt{mk}} \tag{15}$$

Problem 2 Impacts between two blocks and a stationary wall. The first two problems in this Problem Set involve a sequence of individual collisions between two masses. In each collision a stationary mass is impacted by a moving mass, and in each collision it is assumed that there is no loss of energy (the impacts are taken to be perfectly elastic: $e = 1.0$). Since this kind of collision occurs several times, it is worthwhile to establish a general result which can be adapted to any of the collisions which occur in these problems.

For the generic collision, consider that the mass m_a has velocity v_o when it strikes the stationary mass m_b. We set ourselves the problem of predicting the velocities v_a and v_b which the masses m_a and m_b have immediately after the impact. This collision is displayed in the space-time diagram below. Note that in this diagram the slopes of the path lines

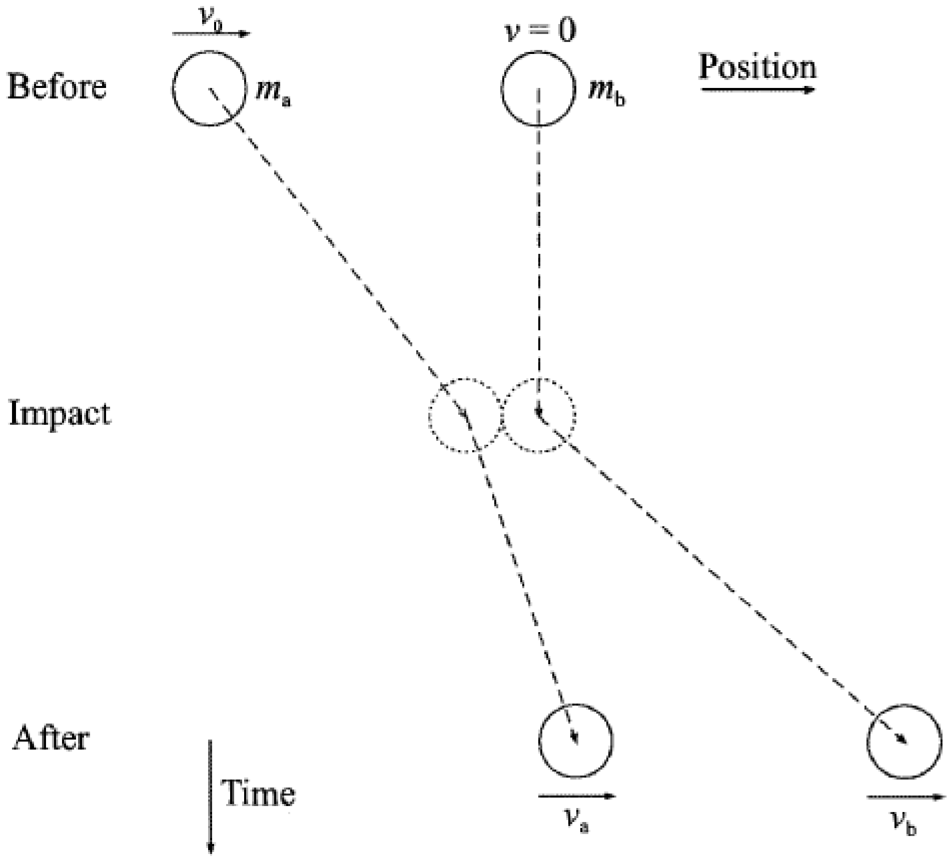

Figure 1: Generic Collision

indicate the velocities of the corresponding masses. The velocities immediately after the impact must satisfy two physical requirements: (i) in perfectly elastic impact the relative velocity of separation $v_b - v_a$ after the impact is equal to the relative velocity of approach v_o before the impact; (ii) the total linear momentum of the system is conserved while it is redistributed between the masses by the impact. These requirements translate into the following pair of equations

$$\begin{aligned} v_b - v_a &= v_o \\ m_a v_a + m_b v_b &= m_a v_o \end{aligned}$$

This is a pair of simultaneous algebraic equations for v_a and v_b whose solution is

$$v_a = \frac{m_a - m_b}{m_a + m_b} v_o \qquad \text{and} \qquad v_b = \frac{2m_a}{m_a + m_b} v_o \tag{1}$$

Note that in this generic solution the initial velocity is taken as positive to the right and that both velocities after the collision are to the right if $m_a > m_b$. If $m_a < m_b$ then v_a is negative which means that m_a rebounds to the left. In the special case where $m_a = m_b$ we have the interesting situation where $v_a = 0$: all the momentum carried by m_a before the impact is transferred to m_b after the impact, leaving m_a motionless.

Now consider the three-mass system of m_1, m_2 (smaller than m_1), and the wall with $m_w = \infty$. Initially the velocity v_{i1} of m_1 is zero as is the velocity v_w of the wall. The initial velocity of m_2 is v_{2i} to the left. In the first impact the moving mass m_2 strikes the stationary mass m_1. The velocities after the impact can be obtained from the generic solution (1) if we identify m_2 with m_a and m_1 with m_b and take the positive direction for velocity to be towards the left with v_{2i} identified with v_o.

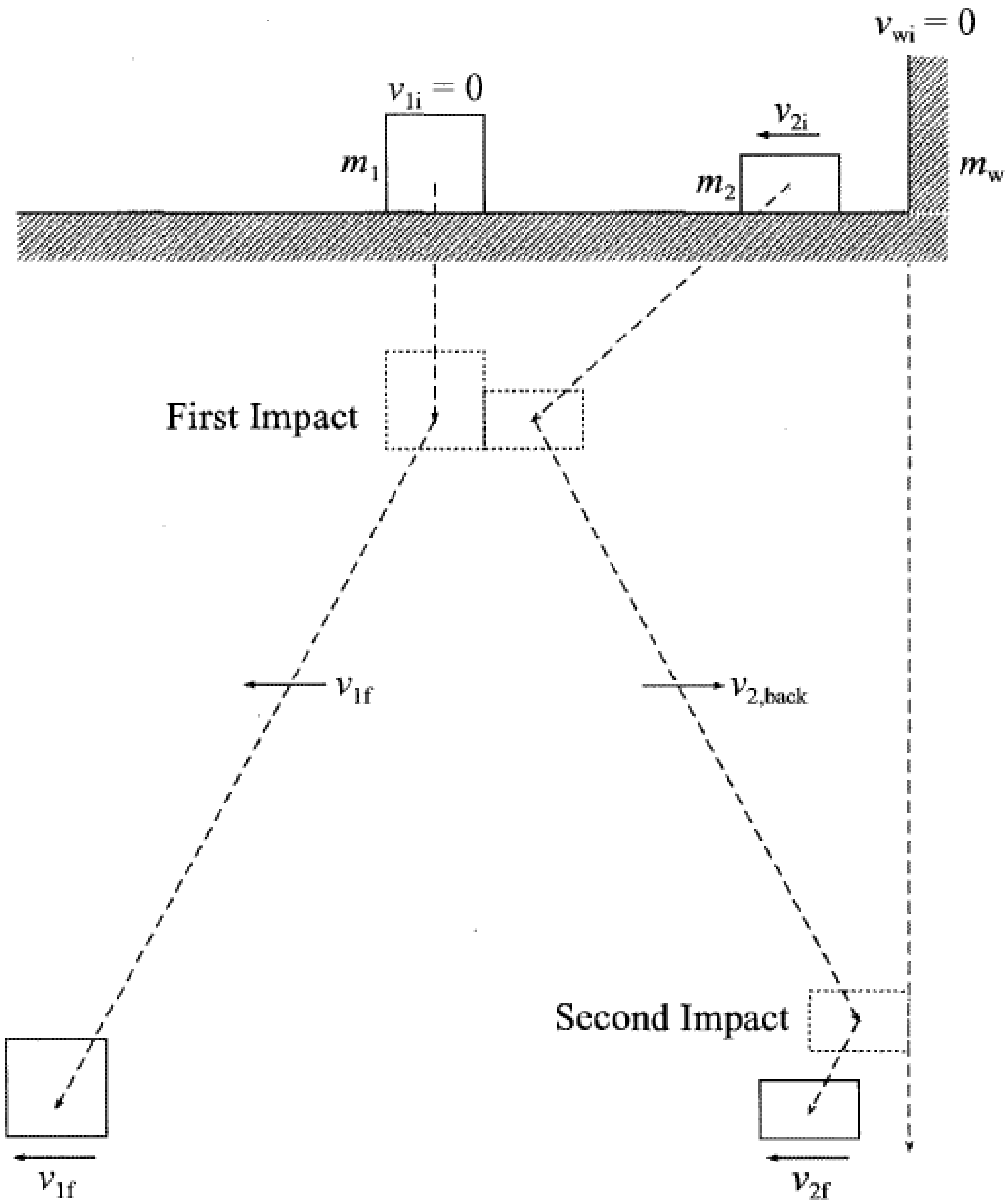

Figure 2: Space-time diagram of sequence of impacts

The final velocity v_{1f} of the mass m_1 after the impact is identified with the velocity v_b of the generic solution, so that from (1)

$$v_{1f} = \frac{2m_2}{m_2 + m_1} v_{21}$$

The velocity $v_{2,back}$ to the right of the mass m_2 after the impact is identified with the velocity $-v_a$ of the generic solution, so that from (1)

$$v_{2,back} = -\frac{m_2 - m_1}{m_2 + m_1} v_{2i} = \frac{m_1 - m_2}{m_1 + m_2} v_{2i}$$

In the second impact the mass m_2 traveling with the velocity $v_{2,back}$ to the right strikes the motionless wall and rebounds with the velocity $v_2 f$ to the left. The magnitude of v_{2f} can be obtained from the generic solution (1) by identifying m_2 with m_a, $m_w = \infty$ with m_b, and $v_{2,back}$ with v_o. Then from (1) we find

$$-v_{2f} = \frac{m_2 - m_w}{m_2 + m_w} v_{2,back} = -v_{2,back} \qquad \text{or} \qquad v_{2f} = v_{2,back}$$

In other words, in the perfectly elastic impact of a finite mass m_2 with an infinite mass, the finite mass rebounds with velocity equal and opposite to the impact velocity.

At this stage we know that the final velocity of m_1 to the left is $v_{1f} = \dfrac{2m_2}{m_2 + m_1} v_{2i}$ and that the final velocity of m_2 to the left is $v_{2f} = v_{2,back} = \dfrac{m_1 - m_2}{m_1 + m_2} v_{2i}$. These two velocities will be equal if

$$v_{1f} = v_{2,back} \qquad \text{or} \qquad 2m_2 = m_1 - m_2 \qquad \text{or} \qquad m_2 = \frac{m_1}{3}$$

Problem 3 Catching a Bullet with a Pendulum. *Momentum* is conserved when the bullet of mass m and velocity v strikes the motionless block of mass M. Then *energy* is conserved as the kinetic energy of the block and bullet, immediately after the impact is transformed into potential energy as the pendulum swings up to the position of maximum height and zero kinetic energy.

When the bullet is caught the block and the bullet have the same velocity V (this implies a coefficient of restitution of zero). The magnitude of the velocity V is obtained by equating the momentum of the bullet plus block immediately before impact ($mv + 0$) to the momentum of the bullet plus block immediately after impact ($mV + MV$) to get

$$V = \frac{m}{M + m} v$$

The kinetic energy of the block plus bullet immediately after impact is

$$KE = \frac{1}{2}(M + m)V^2 = \frac{1}{2} m v^2 \frac{m}{M + m} \qquad (1)$$

The increase in potential energy of the block plus bullet after it has risen the maximum height y is

$$PE = (M + m)gy \qquad (2)$$

The relation between the initial velocity v of the bullet and the height y is obtained by equating the energies (1) and (2) to get

$$y = \frac{v^2}{2g} \left(\frac{m}{M + m} \right)^2$$