2.003J/1.053J Dynamics and Control I, Spring 2007
Professor Peacock
5/14/2007

Lecture 23

Vibrations: Two Degrees of Freedom Systems - Wilberforce Pendulum and Bode Plots

Wilberforce Pendulum

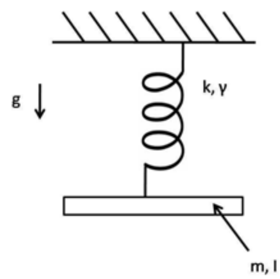

Figure 1: Wilberforce Pendulum. Extension of spring coupled to rotation. Figure by MIT OCW.

m: mass, I: moment of inertia, γ: torsional constant (rotation), k: extension, ϵ: coupling of extension and rotation

Focus on Spring

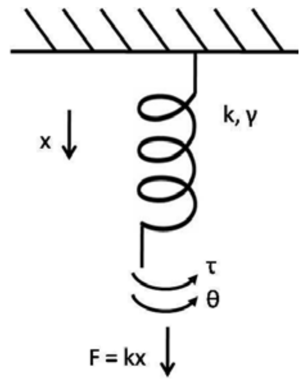

Figure 2: Focus on spring. Figure by MIT OCW.

Force: $F = kx + \frac{1}{2}\epsilon\theta$ (also opposing force in opposite direction)
Torque: $\tau = \gamma\theta + \frac{1}{2}\epsilon x$

$\frac{1}{2}\epsilon\theta$ and $\frac{1}{2}\epsilon x$ represents the coupling for small displacements

Equations of Motion

Can use Lagrange, but we will use momentum principles.

$$m\ddot{x} = mg - kx - \frac{1}{2}\epsilon\theta \tag{1}$$

$$I\ddot{\theta} = -\gamma\theta - \frac{1}{2}\epsilon x \tag{2}$$

Equilibrium Points

$$kx_0 + \frac{1}{2}\epsilon\theta_0 - mg = 0$$

$$\gamma\theta_0 + \frac{1}{2}\epsilon x_0 = 0$$

$$\theta_0 = -\frac{1}{2}\frac{\epsilon}{\gamma}x_0$$

$$x_0 = \frac{mg}{k - \frac{1}{4}\frac{\epsilon^2}{\gamma}}$$

Look at perturbation around (x_0, θ_0)

Let $x = x_0 + z$, $\theta = \theta_0 + \phi$

Substitute in Equation (1) and (2)

$$m\ddot{z} + kz + \frac{1}{2}\epsilon\phi + kx_0 + \frac{1}{2}\epsilon\theta_0 = mg$$

$$I\ddot{\phi} + \gamma\phi + \frac{1}{2}\epsilon z + \gamma\theta_0 + \frac{1}{2}\epsilon x_0 = 0$$

$\gamma\theta_0 + \frac{1}{2}\epsilon x_0 = 0$ and $kx_0 + \frac{1}{2}\epsilon\theta_0 = mg$ are restatements of the equilibrium conditions, so we can remove those terms to obtain

$$m\ddot{z} + kz + \frac{1}{2}\epsilon\phi = 0$$

$$I\ddot{\phi} + \gamma\phi + \frac{1}{2}\epsilon z = 0.$$

$$\begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix} \begin{Bmatrix} \ddot{z} \\ \ddot{\phi} \end{Bmatrix} + \begin{bmatrix} k & \frac{1}{2}\epsilon \\ \frac{1}{2}\epsilon & \gamma \end{bmatrix} \begin{Bmatrix} z \\ \phi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Mass Matrix or Inertia Matrix:

$$\begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix}$$

Stiffness Matrix:

$$\begin{bmatrix} k & \frac{1}{2}\epsilon \\ \frac{1}{2}\epsilon & \gamma \end{bmatrix}$$

Free Response Solution

Free Response:

$$\begin{Bmatrix} z \\ \phi \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} \cos(\omega t - \psi)$$

Substituting the free response solution into the system of equations gives:

$$\underbrace{\begin{bmatrix} k - m\omega^2 & \frac{1}{2}\epsilon \\ \frac{1}{2}\epsilon & \gamma - I\omega^2 \end{bmatrix}}_{\text{det}=0} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3}$$

Thus:

$$(k - m\omega^2)(\gamma - I\omega^2) - (\frac{1}{2}\epsilon)^2 = 0$$

$$(\frac{k}{m} - \omega^2)(\frac{\gamma}{I} - \omega^2) = (\frac{1}{2}\frac{\epsilon}{\sqrt{mI}})^2$$

Define:

$$\frac{k}{m} = \frac{\gamma}{I} = \omega_*^2$$

$$(\omega_*^2 - \omega^2) = \pm\frac{1}{2}\frac{\epsilon}{\sqrt{mI}}$$

$$\omega_1^2 = \omega_*^2 + \frac{\epsilon}{2\sqrt{mI}}$$

$$\omega_2^2 = \omega_*^2 - \frac{\epsilon}{2\sqrt{mI}}$$

These are two natural frequencies of oscillation.

Mode Shapes

ω_1^2:

$$(k - m\omega_1^2)c_1 + \frac{1}{2}\epsilon c_2 = 0$$

$$(\frac{k}{m} - \omega_1^2)c_1 + \frac{1}{2}\frac{\epsilon}{m}c_2 = 0$$

$$(\omega_*^2 - \omega_1^2)c_1 + \frac{1}{2}\frac{\epsilon}{m}c_2 = 0$$

$$(-\frac{\epsilon}{2\sqrt{mI}})c_1 + \frac{1}{2}\frac{\epsilon}{m}c_2 = 0$$

$$\boxed{\frac{c_2}{c_1} = \sqrt{\frac{m}{I}}}$$

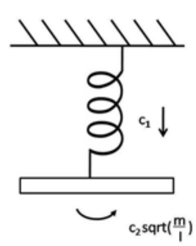

Figure 3: If you change x, θ increases by a prescribed amount. Fixed by that ratio. Figure by MIT OCW.

ω_2^2:

$$(k - m\omega_2^2)c_1 + \frac{1}{2}\epsilon c_2 = 0$$

$$(\frac{\epsilon}{2\sqrt{mI}})c_1 + \frac{1}{2}\frac{\epsilon}{m}c_2 = 0$$

$$\boxed{\frac{c_2}{c_1} = -\sqrt{\frac{m}{I}}}$$

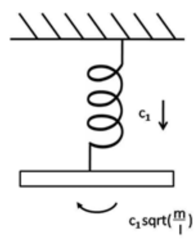

Figure 4: If you change x, θ increases in the other direction based on the ratio. Figure by MIT OCW.

General Solution for Free Response

$$\begin{Bmatrix} z \\ \phi \end{Bmatrix} = \alpha_1 \begin{Bmatrix} 1 \\ \sqrt{\frac{m}{I}} \end{Bmatrix} \cos(\omega_1 t - \psi_1) + \alpha_2 \begin{Bmatrix} 1 \\ -\sqrt{\frac{m}{I}} \end{Bmatrix} \cos(\omega_2 t - \psi_2)$$

α and ψ are set by initial conditions.

At $t = 0$ s:

$$\begin{Bmatrix} z \\ \phi \end{Bmatrix} = \begin{Bmatrix} A \\ 0 \end{Bmatrix} \qquad \begin{Bmatrix} \dot{z} \\ \dot{\phi} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Stretch spring by a distance A from resting point

$$\begin{Bmatrix} A \\ 0 \end{Bmatrix} = \alpha_1 \begin{Bmatrix} 1 \\ \sqrt{\frac{m}{I}} \end{Bmatrix} \cos(\psi_1) + \alpha_2 \begin{Bmatrix} 1 \\ -\sqrt{\frac{m}{I}} \end{Bmatrix} \cos(\psi_2) \tag{4}$$

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \omega_1\alpha_1 \begin{Bmatrix} 1 \\ \sqrt{\frac{m}{I}} \end{Bmatrix} \sin(\psi_1) + \omega_2\alpha_2 \begin{Bmatrix} 1 \\ -\sqrt{\frac{m}{I}} \end{Bmatrix} \sin(\psi_2) \tag{5}$$

Using Equation (5):

$$\omega_1\alpha_1 \sin\psi_1 + \omega_2\alpha_2 \sin\psi_2 = 0$$

$$\sqrt{\frac{m}{I}}\omega_1\alpha_1\sin\psi_2 - \sqrt{\frac{m}{I}}\omega_2\alpha_2\sin\psi_2 = 0$$

$$\boxed{\psi_1 = \psi_2 = 0}$$

Using Equation (4) with $\psi_1 = \psi_2 = 0$:

$$\boxed{\alpha_1 = \alpha_2 = \frac{A}{2}}$$

$$\boxed{\left\{\begin{array}{c} z \\ \phi \end{array}\right\} = \frac{A}{2}\left\{\begin{array}{c} 1 \\ \sqrt{\frac{m}{I}} \end{array}\right\}\cos\omega_1 t + \frac{A}{2}\left\{\begin{array}{c} 1 \\ -\sqrt{\frac{m}{I}} \end{array}\right\}\cos\omega_2 t}$$

α_1 and α_2 are the same. Means the two modes are weighted equally.

The equal weighting implies that when the spring oscillates there is accompanying rotation. Sometimes the spring-mass will be still; sometimes there will be spring extension without rotation; sometimes there will be rotation (both directions) with no spring extension.

Analysis For Weak Spring Extension-Rotation Coupling

$$\cos\alpha + \cos\beta = 2\cos(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$$

$$\cos\alpha - \cos\beta = 2\sin(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$$

$$z = 2\frac{A}{2}\cos(\frac{\omega_1+\omega_2}{2}t)\cos(\frac{\omega_1-\omega_2}{2}t)$$

$\epsilon << 1$: Weak Coupling

$$\omega_1 = \sqrt{\omega_*^2 + \frac{\epsilon}{2\sqrt{mI}}} = \omega_*\sqrt{1 + \frac{\epsilon}{2\omega_*^2\sqrt{mI}}}$$

$$\omega_1 \cong \omega_* + \frac{\epsilon}{4\omega_*\sqrt{mI}}$$

$$\omega_2 \cong \omega_* - \frac{\epsilon}{4\omega_*\sqrt{mI}}$$

Therefore:

$$z = A\cos(\omega_* t)\cos(\frac{\epsilon}{4\omega_*\sqrt{mI}}t)$$

$$\phi = A\sqrt{\frac{m}{I}}\sin(\omega_* t)\sin(\frac{\epsilon}{4\omega_*\sqrt{mI}}t)$$

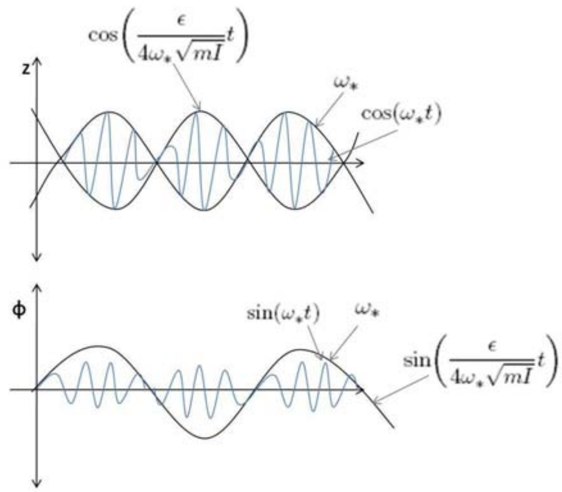

Figure 5: Example of beating. The rotation is out of phase with the extension. Figure by MIT OCW.

Beating

Bode Plots

Response of a 1 degree of freedom system

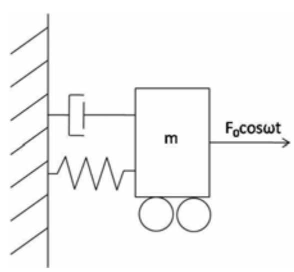

Figure 6: Cart with spring and dashpot attached. Figure by MIT OCW.

$$\frac{x}{F_0/k} = \frac{1}{\sqrt{(1 - \frac{\omega^2}{\omega_n^2}) - (2\zeta\frac{\omega}{\omega_n})^2}}$$

Pole:
Value of ω that sets denominator to 0.

Zeros:
Values of ω that set to numerator to 0. (None in this example)

Express the *magnification* $M = \left|\frac{x}{F_0/k}\right|$ in decibels (dB).

$$20\log_{10} M = \text{ decibels of } M.$$

Small frequency ($\omega = 0$) $\Rightarrow 20\log_{10} 1 = 0$

High frequency $\frac{\omega}{\omega_n} >> 1 \Rightarrow 20\log_{10}(\frac{1}{\omega^2/\omega_n^2}) = 20\log\left(\frac{\omega}{\omega_n}\right)^{-2} = -40\log\frac{\omega}{\omega_n}$

-40 dB/decade of frequency

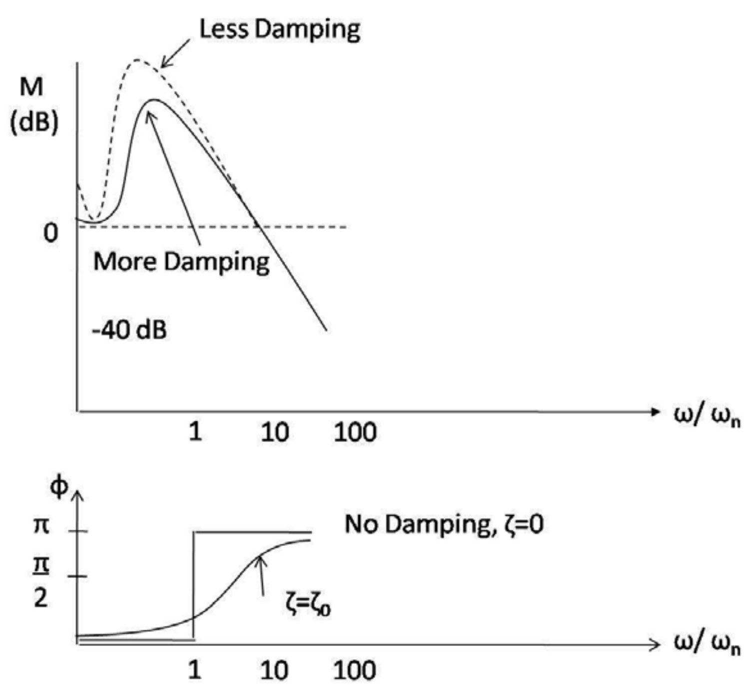

Figure 7: Bode plot of modeled system. Figure by MIT OCW.

With little damping in system, where is maximum? A little below natural frequency ω_n.

For more information, see
`http://www.swarthmore.edu/NatSci/echeeve1/Ref/LPSA/Bode/Bode.html`