
These problems are related to the material covered in Lectures 21-22. I have made every effort to proof-read them, but some errors may remain. The first person to spot each error will receive 1-5 points of extra credit.

The problem set is due by the start of class on 12/3/2013 and should be submitted electronically as a pdf-file e-mailed to the instructor. You can use the latex source for this problem set as a template for writing up your solutions; be sure to include your name in your solutions and to identify collaborators and any sources not listed in the syllabus.

Recall that we have defined a *curve* as a smooth projective variety of dimension one (and varieties are defined to be irreducible algebraic sets).

Problem 1. Bezout's theorem (50 points)

In this problem k is an algebraically closed field.

A curve in \mathbb{P}^2 is called a *plane curve*.¹

- (a) Prove that every plane curve X/k is a hypersurface, meaning that its ideal $I(X)$ is of the form (f) , where f is a homogeneous polynomial in $k[x, y, z]$. Then show that every generator for $I(X)$ has the same degree.

The *degree* of X (denoted $\deg X$) is the degree of any generator for its homogeneous ideal.

- (b) Let F/k be a function field, let P be a place of F , and let $f \in \mathcal{O}_P$. Prove that the ring $\mathcal{O}_P/(f)$ is a k -vector space of dimension $\text{ord}_P(f)$.

Given a nonconstant homogeneous polynomial $g \in k[x, y, z]$ that is relatively prime to f , we can represent g as an element of the local ring $\mathcal{O}_{X,P}$ of functions in X that are regular at P by picking a homogeneous polynomial h that does not vanish at P and representing g as g/h reduced modulo $I(X)$, an element of $k(X)$. Note that in terms of computing $\text{ord}_P(g)$ it makes no difference which h we pick, $\text{ord}_P(g)$ will always be equal to the order of vanishing of g at P , a nonnegative integer. We then define the *divisor* of g in $\text{Div}_k X$ to be

$$\text{div}_X g = \sum \text{ord}_P(g)P.$$

Note that $\text{div}_X g$ is *not* a principal divisor.² Indeed, $\deg \text{div}_X g$ is never zero.

- (c) Prove that $\deg \text{div}_X g$ depends only on $\deg g$ (i.e. $\deg \text{div}_X g = \deg \text{div}_X h$ whenever g and h have the same degree and are both relatively prime to f). Then prove that $\deg \text{div}_X g$ is a linear function of $\deg g$.

Now suppose that g is irreducible and nonsingular, so it defines a plane curve Y/k .

- (d) Prove that $\deg \text{div}_Y f = \deg \text{div}_X g$.

¹Plane curves are not usually required to be smooth or irreducible, but ours are.

²By varying h locally we eliminate the poles that would be present if we fixed a global choice for h .

Definition 1. Let f and g be two nonconstant homogeneous polynomials in $k[x, y, z]$ with no common factor, and let P be a point in \mathbb{P}^2 . The *intersection number* of f and g at P is

$$I_P(f, g) := \dim_k \mathcal{O}_{\mathbb{P}^2, P} / (f, g)$$

Here $\mathcal{O}_{\mathbb{P}^2, P}$ denotes the ring of functions in $k(\mathbb{P}^2)$ that are regular at P , and f and g are represented as elements of this ring by choosing homogeneous denominators of appropriate degree that do not vanish at P , exactly as described above.

As above, let X/k and Y/k denote plane curves defined by relatively prime homogeneous polynomials f and g , and let $I(f, g) = \sum_P I_P(f, g)$.

(e) Prove that $I(f, g)$ is equal to $\deg \operatorname{div}_X g = \deg \operatorname{div}_Y f$.

(f) Prove **Bezout's Theorem** for plane curves:

$$I(f, g) = \deg f \deg g.$$

In fact Bezout's theorem holds even when f and g are not necessarily irreducible and nonsingular, but you need not prove this. It should be clear that f and g do not need to be irreducible; just factor them and apply the theorem to all pairs of factors. Your proof should also handle cases where just one of f or g is singular; it takes a bit more work to handle the case where both f and g are singular and intersect at a common singularity. The assumption that $k = \bar{k}$ is necessary, in general, but the inequality $I(f, g) \leq \deg f \deg g$ always holds.

Problem 2. Derivations and differentials (50 points)

A *derivation* on a function field F/k is a k -linear map $\delta: F \rightarrow F$ such that

$$\delta(fg) = \delta(f)g + f\delta(g).$$

for all $f, g \in F$.

(a) Prove that the following hold for any derivation δ on F/k :

- (i) $\delta(c) = 0$ for all $c \in k$.
- (ii) $\delta(f^n) = n f^{n-1} \delta(f)$ for all $f \in F^\times$ and $n \in \mathbb{Z}$.
- (iii) If k has positive characteristic p then $\delta(f^p) = 0$ for all $f \in F$.
- (iv) $\delta(f/g) = (\delta(f)g - f\delta(g))/g^2$ for all $f, g \in F$ with $g \neq 0$.

To simplify matters, we henceforth assume that k has characteristic zero.³

The simplest example of a derivation is in the case where $F = k(x)$ is the rational function field and $\delta: F \rightarrow F$ is the map defined by $\delta(f) = \partial f / \partial x$. We want to generalize this example to arbitrary function fields.

³For those who are interested, the key thing that changes in characteristic $p > 0$ is that everywhere we require an element x to be transcendental we need to additionally require it to be a *separating element*, which means that $F/k(x)$ is a separable extension.

Let x be a transcendental element of F/k . Any $y \in F$ is then algebraic over $k(x)$ and has a minimal polynomial $\lambda \in k(x)[T]$. After clearing denominators we can assume that $\lambda \in k[x, T]$. We now formally define

$$\frac{\partial y}{\partial x} := -\frac{\partial \lambda / \partial x}{\partial \lambda / \partial T}(y) \in k(x, y) \subseteq F$$

and let the map $\delta_x: F \rightarrow F$ send y to $\partial y / \partial x$.

One can show (but you are not asked to do this) that δ_x is a derivation on F/k . Note that we get a derivation δ_x for each transcendental x in F . Now let D_F be the set of all derivations on F/k .

(b) Let x be a transcendental element of F/k . Prove that for any $\delta_1, \delta_2 \in D_F$ we have $\delta_1(x) = \delta_2(x) \Rightarrow \delta_1 = \delta_2$. Conclude that δ_x is the unique $\delta \in D_F$ for which $\delta(x) = 1$.

(c) Prove the following:

- (i) For all $\delta_1, \delta_2 \in D_F$ the map $(\delta_1 + \delta_2): F \rightarrow F$ defined by $f \mapsto \delta_1(f) + \delta_2(f)$ is a derivation (hence an element of D_F).
- (ii) For all $f \in F$ and $\delta \in D_F$ the map $(f\delta): F \rightarrow F$ defined by $g \mapsto f\delta(g)$ is a derivation (hence an element of D_F).
- (iii) Every $\delta \in D_F$ satisfies $\delta = \delta(x)\delta_x$ (in particular, the *chain rule* $\delta_y = \delta_y(x)\delta_x$ holds for any transcendental $x, y \in F/k$).

It follows that we may view D_F as one-dimensional F -vector space with any δ_x as a basis vector. But rather than fixing a particular basis vector; instead, let us define a relation on the set S of pairs (u, x) with $u, x \in F$ and x transcendental over k :

$$(u, x) \sim (v, y) \iff v = u\delta_y(x). \tag{1}$$

(d) Prove that \sim is an equivalence relation on S .

For each transcendental element $x \in F/k$, let the symbol dx denote the equivalence class of $(1, x)$, and for $u \in F$ define udx to be the equivalence class of (u, x) ; we call dx a *differential*. It follows from part (iii) of (d) that every derivation δ can be uniquely represented as $\delta = udx$ for some $u \in F$, but now we have the freedom to change representations; we may also write $\delta = vdy$ for any transcendental element y , where $v = u\delta_y(x) = u\partial x / \partial y$.

(e) Prove that $d(x+y) = dx + dy$ and $d(xy) = xdy + ydx$ for all transcendental $x, y \in F/k$.

Let us now extend our differential notation to elements of F that are not transcendental over k . Recall that k is algebraically closed in F , so we only need to consider elements of k .

(f) Prove that defining $da = 0$ for all $a \in k$ ensures that (e) holds for all $x, y \in F$, and that no other choice does.

Now momentarily forget everything above and just define Δ_F to be the F -vector space generated by the set of formal symbols $\{dx : x \in F\}$, subject to the relations

$$(1) \ d(x+y) = dx + dy, \quad (2) \ d(xy) = xdy + ydx, \quad (3) \ da = 0 \text{ for } a \in k.$$

Note that x and y denote elements of F (functions), not free variables, so Δ_F reflects the structure of F and will be different for different function fields.

(g) Prove that $\dim_F \Delta_F = 1$, and that any dx with $x \notin k$ is a basis.

The set $\Delta = \Delta_F$ is often used as an alternative to the set of Weil differentials Ω . They are both one-dimensional F -vector spaces, hence isomorphic (as F -vector spaces). But in order to be useful, we need to associate divisors to differentials in Δ , as we did for Ω .

For any differential $\omega \in \Delta$ and any place P , we may pick a uniformizer t for P and write $\omega = wdt$ for some function $w \in F$ that depends on our choice of t ; note that t is necessarily transcendental over k , since it is a uniformizer. We then define $\text{ord}_P(\omega) := \text{ord}_P(w)$, and the divisor of ω is then given by

$$\text{div } \omega := \sum_P \text{ord}_P(\omega)P.$$

As in Problem 1, the value $\text{ord}_P(\omega)$ does not depend on the choice of the uniformizer t .

(h) Prove that $\text{div } u dv = \text{div } u + \text{div } dv$ for any $u, v \in F$. Conclude that the set of nonzero differentials in Δ constitutes a linear equivalence class of divisors.

(i) Let $F = k(t)$ be the rational function field. Compute $\text{div } dt$ and prove that it is a canonical divisor. Conclude that a divisor $D \in \text{Div}_k C$ is canonical if and only if $D = \text{div } df$ for some transcendental $f \in F$.

Part (i) holds for arbitrary curves, but you are not asked to prove this. It follows that the space of differentials Δ plays the same role as the space of Weil differentials Ω , and it has the virtue of making explicit computations much easier.

(j) Prove that the curve $x^2 + y^2 + z^2$ over \mathbb{Q} has genus 0 (even though it is not isomorphic to \mathbb{P}^1 because it has no rational points) by explicitly computing a canonical divisor.

Problem 3. Survey

Complete the following survey by rating each problem on a scale of 1 to 10 according to how interesting you found the problem (1 = “mind-numbing,” 10 = “mind-blowing”), and how difficult you found the problem (1 = “trivial,” 10 = “brutal”). Also estimate the amount of time you spent on each problem.

	Interest	Difficulty	Time Spent
Problem 1			
Problem 2			

Please rate each of the following lectures that you attended, according to the quality of the material (1=“useless”, 10=“fascinating”), the quality of the presentation (1=“epic fail”, 10=“perfection”), the pace (1=“way too slow”, 10=“way too fast”), and the novelty of the material (1=“old hat”, 10=“all new”).

Date	Lecture Topic	Material	Presentation	Pace	Novelty
11/19	Riemann’s inequality				
11/21	The Riemann-Roch theorem				

Feel free to record any additional comments you have on the problem sets or lectures; in particular, how you think they might be improved.

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