# 18.725: EXERCISE SET 8

DUE THURSDAY NOVEMBER 6

(1) Assume the characteristic of $k$ is not 2. Let $b : X \to \mathbb{A}^2$ be the blow–up of $\mathbb{A}^2$ at the origin $(0,0)$, and let $U = \mathbb{A}^2 - \{(0,0)\}$.

(i) Show that the morphism $b^{-1}(U) \to U$ is an isomorphism.

(ii) If $Z \subset \mathbb{A}^2$ is a closed subset not equal to $\{(0,0)\}$, the *strict transform* of $Z$ is defined to be the closure of $Z \cap U$ in $X$, where $Z \cap U$ is viewed as a subset of $X$ via the isomorphism in (i). Compute the strict transform of $V(xy) \subset \mathbb{A}^2$.

(iii) Compute the strict transform of $V(y^2 - x^2(x+1)) \subset \mathbb{A}^2$.

(2) Let $X$ be a variety and $x \in X$ a point. Denote by $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ the maximal ideal of the local ring $\mathcal{O}_{X,x}$. Since the residue field of $\mathcal{O}_{X,x}$ is $k$, the quotient $\mathfrak{m}/\mathfrak{m}^2$ is a $k$–vector space. Show that

$$\dim_k(\mathfrak{m}/\mathfrak{m}^2) \geq \dim(X).$$

Show that if $\mathcal{O}_{X,x}$ is a regular local ring, then this in fact is an equality.

(3) Let $f : X \to Y$ be a non–constant finite morphism of varieties with $\dim(X) = \dim(Y) = 1$. Fix a point $y \in Y$ and let $x_1, \ldots, x_r$ be the points in $f^{-1}(y)$. Assume that the local rings $\mathcal{O}_{Y,y}$ and $\{\mathcal{O}_{X,x_i}\}_{i=1}^r$ are all regular. For each $x_i$, let $e(x_i)$ denote the dimension of the $k$–vector space $\mathcal{O}_{X,x_i}/\mathfrak{m}_y\mathcal{O}_{X,x_i}$. Show that

$$[k(X) : k(Y)] = \sum_{i=1}^{r} e(x_i).$$

(4) A morphism of varieties $f : X \to Y$ is said to be *projective* if for some $n$ there is a factorization of $f$

$$X \xrightarrow{j} \mathbb{P}^n \times Y \xrightarrow{p_2} Y,$$

where $j$ identifies $X$ with an irreducible closed subvariety of $\mathbb{P}^n \times Y$.

(i) Show that a projective morphism is closed.

(ii) If $Y$ is affine, show that any finite morphism $f : X \to Y$ is projective.

(5) Suppose the characteristic of $k$ is $p > 0$. On an earlier homework, we saw an example of a morphism of varieties $f : X \to Y$ which was not an isomorphism but was a homeomorphism on the underlying topological spaces.

(i) Let $Y$ be an affine variety, and let $f : X \to Y$ be a morphism of varieties whose map on underlying spaces is a homeomorphism. Show that $X$ is affine.

(ii) Let $R$ be the coordinate ring of $Y$, and let $X$ be a second affine variety with coordinate ring $S$. Let $f : X \to Y$ be a morphism associated to a map of rings $f^* : R \to S$. Give

*Date*: October 30, 2003.

necessary and sufficient conditions on the map $f^*$ for the morphism $f$ to be a homeomorphism on the underlying topological spaces.

(6) Let $X \subset \mathbb{A}^3$ be the zero locus of $z^2 - xy$.

(i) Show that $\dim(X) = 2$.

(ii) Find a closed subvariety $W \subset X$ of codimension 1 which is not of the form $V(g)$ for some $g \in \Gamma(X, \mathcal{O}_X)$.

(7) Fix positive integers $N$ and $r$, and let

$$F : (\text{varieties}) \longrightarrow (\text{Set})$$

be the contravariant functor which to any variety $Y$ associates the set of polynomials in $\Gamma(Y, \mathcal{O}_Y)[X_1, \dots, X_r]$ of degree $N$. If $g : W \to Y$ is a morphism of varieties, then the map $F(Y) \to F(W)$ is the one induced by the map

$$\Gamma(Y, \mathcal{O}_Y)[X_1, \dots, X_r] \longrightarrow \Gamma(W, \mathcal{O}_W)[X_1, \dots, X_r]$$

induced by $g^*$. Show that $F$ is representable. In other words, there exists a variety $M$ and an isomorphism of functors $h_M \simeq F$, where $h_M$ is the functor sending $Y$ to $\mathrm{Hom}(Y, M)$.