18.725: EXERCISE SET 1

DUE TUESDAY SEPTEMBER 16

(1) The goal of this exercise is to prove a stronger version of Noether's normalization lemma in the case when the ground field k is infinite. So suppose A is a finitely generated k–algebra and choose generators $x_1, \ldots, x_n \in A$ for A. Prove that there exists $y_1, \ldots, y_r \in A$ which are *linear* combinations of the x_i such that A is integral over $k[y_1, \ldots, y_r]$. Geometrically this corresponds to a projection from the variety in $\mathbb{A}^n$ defined by A onto a linear subspace of $\mathbb{A}^n$.

Hint: Order the x_i so that $x_1, \ldots, x_r$ are algebraically independent and each of $x_{r+1}, \ldots, x_n$ are algebraic over $k[x_1, \ldots, x_r]$. Then proceed by induction on r. If $n = r$ there is nothing to prove, so suppose $n > r$ and that the result holds for $n - 1$. Since x_n is algebraic over $k[x_1, \ldots, x_{n-1}]$, there exists polynomial f in n variables such that $f(x_1, \ldots, x_n) = 0$. Let F be the homogeneous part of f of highest degree. Show that there exists $\lambda_1, \ldots, \lambda_{n-1} \in k$ such that $F(\lambda_1, \ldots, \lambda_{n-1}, 1) \neq 0$. Then let $x_i' = x_i - \lambda_i x_n$, and show that x_n is integral over $k[x_1', \ldots, x_{n-1}']$.

(2) Is the ring $A = k[T_1, T_2]/(T_1^2 - T_2^3)$ integrally closed in its field of fractions? Prove or disprove.

(3) Show that the set of pairs $(a, b) \in k^2$ for which either a or b is zero is an algebraic set. What are its irreducible components?

(4) Suppose the characteristic of k is not 2. Find the irreducible components of the affine algebraic set defined by the equations $X_1^2 + X_2^2 + X_3^2 = 0$, $X_1^2 - X_2^2 - X_3^2 + 1 = 0$. What happens in characteristic 2?

(5) Same as (4) for the equations $X_2^2 - X_1X_3 = 0$, $X_1^2 - X_2^3 = 0$.

(6) Show that the set of invertible $n \times n$ matrices $GL_n(k)$ is naturally a closed algebraic subset of k^{n^2+1}. What about the orthogonal group $O(n, k)$ consisting of invertible matrices A for which $A^T = A^{-1}$?

(7) Give an example of two algebraic sets V and V' in k^n (some n) such that $I(V) \cdot I(V') \neq I(V \cup V')$.

(8) Find the radical in $k[x, y]$ of the ideal generated by x^2y^3 and x^5y.

(9) Is the set $\{(z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\}$ in $\mathbb{C}^2$ an algebraic subset?

(10) Describe the algebraic set defined by the equations $X^2+Y^2+Z^2 = 1$, $X^2+Y^2-Y = 0$, and $X - Z$.

Date: September 4, 2003.