

18.702 Homework for Wednesday, February 16, corrected

First version omitted the very important requirement $d(\gamma_0) = 1$ from the definition of a harmonic graph.

1. Chapter 10, Exercise 6.4 (*characters of the octahedral group*).
2. Chapter 10, Exercise 6.9 (*identify simple groups from character table*).
3. A connected finite graph is called *harmonic* if it has the extra structure of
 - 1) a positive integer weight $d(\gamma)$ labeling each vertex γ ; and
 - 2) a special vertex called γ_0 , with $d(\gamma_0) = 1$
 subject to the following requirement: for every vertex γ ,

$$2d(\gamma) = \sum_{\text{edges } \{\gamma, \gamma'\}} d(\gamma').$$

That is, the label of each vertex is half the sum of the labels of adjacent vertices. The problem is to classify all possible harmonic graphs. (Here is one family of examples, to get you started. Fix an integer $N \geq 3$. Let the vertices be the vertices of a regular N -gon, and the edges the usual edges. Label each vertex with a 1, and pick one vertex to call γ_0 . Since each vertex has exactly two adjacent vertices, it's easy to see that this is a harmonic graph.)

4. Suppose G is a finite subgroup of SU_2 . Write (τ, \mathbb{C}^2) for the defining representation of G ; the τ stands for "tautological." Now make a graph having one vertex γ for every (equivalence class of) irreducible representation (γ, V_γ) of G . There is a distinguished vertex γ_0 corresponding to the trivial representation of G .

Join the vertex γ' to γ by $e(\gamma', \gamma)$ (directed) edges if the irreducible representation γ' appears $e(\gamma', \gamma)$ times in the representation $\text{Hom}(\tau, \gamma)$.

- a) Show that $e(\gamma', \gamma)$ is equal to

$$\langle \chi_{\gamma'}, \overline{\chi_\tau} \chi_\gamma \rangle = |G|^{-1} \sum_{g \in G} \overline{\chi_{\gamma'}(g)} \overline{\chi_\tau(g)} \chi_\gamma(g).$$

- b) Show that $e(\gamma', \gamma) = e(\gamma, \gamma')$; so we can think of the edges as undirected.
- c) Label each vertex γ by $d(\gamma) = \dim V_\gamma$. Show that the graph is harmonic. (The hard part is showing that the graph is connected; I haven't really explained how you might prove that. So don't worry if that's the only part you can't figure out.)
- d) In section 6.12 you studied finite subgroups of SO_3 . In section 9.4 you studied a two-to-one map from SU_2 to SO_3 . Every finite subgroup F of SO_3 has a preimage \tilde{F} in SU_2 having twice as many elements, and called the *binary something or other*. For example, the preimage \tilde{I} of the icosahedral group is called the *binary icosahedral group*; it has 120 elements. In your list of harmonic graphs from problem 3, decide which ones correspond to the binary groups \tilde{T} , \tilde{O} , and \tilde{I} (of orders 24, 48, and 120).

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