NAME: SOLUTIONS

18.075 In–class Exam # 1
Wednesday, September 29, 2004

Justify your answers. Cross out what is not meant to be part of your final answer. Total number of points: 45.

I. (5 pts) Show that for any complex numbers z_1 and z_2,

$$||z_1| - |z_2|| \leq |z_1 + z_2|.$$

It suffices to show that

$$\left||z_1| - |z_2|\right|^2 \leq |z_1 + z_2|^2 = (\bar{z}_1 + \bar{z}_2) \cdot (z_1 + z_2) = |z_1|^2 + |z_2|^2 + 2\mathrm{Re}(\bar{z}_1 z_2)$$

The LHS equals

$$\left||z_1| - |z_2|\right|^2 = |z_1|^2 + |z_2|^2 - 2|z_1 z_2| = |z_1|^2 + |z_2|^2 - 2|\bar{z}_1 \cdot z_2|.$$

Hence, we have to show that

$$\cancel{|z_1|^2} + \cancel{|z_2|^2} - 2|\bar{z}_1 \cdot z_2| \leq \cancel{|z_1|^2} + \cancel{|z_2|^2} + 2\,\mathrm{Re}(\bar{z}_1 \cdot z_2)$$

$$\Leftrightarrow \quad |\bar{z}_1 \cdot z_2| \geq -\mathrm{Re}(\bar{z}_1 \cdot z_2).$$

Let $w = \bar{z}_1 \cdot z_2$: $\quad |w| \geq -\mathrm{Re}\,w$.

If $w = u + iv$, u, v: real, then we have to show that $\sqrt{u^2+v^2} \geq -u$

For $u > 0$, this statement is obviously true.

For $u \leq 0$, $0 \leq -u \leq \sqrt{u^2+v^2} \Leftrightarrow u^2 \leq u^2+v^2 \Leftrightarrow v^2 \geq 0$: true.

Hence, we proved that $\left||z_1| - |z_2|\right| \leq |z_1| + |z_2|$.

II. (5 pts) Find all possible values of

$$(-\sqrt{3}+i)^{1/5}.$$

Let $z = -\sqrt{3} + i = r \cdot e^{i\theta_P}$; $\quad -\pi < \theta_P \leq \pi$.

$r = |z| = \sqrt{3+1} = 2$

$\theta_p = \arctan \frac{1}{-\sqrt{3}} = \begin{cases} -\pi/6 \\ \pi - \pi/6 = \frac{5\pi}{6} \end{cases}$, of which we

take $\theta_p = \frac{5\pi}{6}$ because z lies in the 2nd quadrant.

$$z^{1/5} = \left(2 e^{i\frac{5\pi}{6} + i2k\pi}\right)^{1/5} = \underbrace{\sqrt[5]{2}}_{>0} \cdot e^{i\frac{\pi}{6} + i\frac{2k}{5}\pi},$$

where $\quad k = 0, 1, 2, 3, 4.$

III.

1. (3 pts) Can the function $u(x,y) = x^2 - y^2 - x - y$ be the REAL part of an analytic function $f(z) = u(x,y) + iv(x,y)$? **Hint:** You may use the Laplace equation, if you wish.

We check whether $u(x,y)$ satisfies the Laplace eqn.

$$\frac{\partial^2 u}{\partial x^2} = 2, \qquad \frac{\partial^2 u}{\partial y^2} = -2$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

So, yes, it is possible that u is the real part of an analytic function.

2. (5 pts) Determine all functions $v(x,y)$ such that $f(z) = u(x,y) + iv(x,y)$ is analytic.

We apply the Cauchy-Riemann eqs: $\underbrace{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}_{①}, \quad \underbrace{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}_{②}$

①: $\frac{\partial v}{\partial y} = 2x - 1 \Rightarrow v(x,y) = 2xy - y + C_1(x)$

②: $-2y - 1 = -2y - C_1'(x) \Leftrightarrow C_1'(x) = 1 \Leftrightarrow C_1(x) = x + K$

real const.

So, $v(x,y) = 2xy - y + x + K$

3. (3 pts) Find explicitly as a function of z the $f(z)$ such that

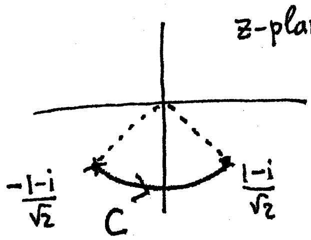
$$f(z) = u(x, y) + iv(x, y).$$

$$\begin{aligned} f(z) &= x^2-y^2-x-y+i(2xy-y+x+K) \\ &= (x^2-y^2+i\,2xy)-(x+iy)+i(x+iy)+iK \\ &= (x+iy)^2+(-1+i)(x+iy)+iK \\ &= z^2+(-1+i)z+iK, \qquad K: \text{real const.} \end{aligned}$$

IV. (6 pts) Compute the line integral

$$\int_C \frac{(z^3+z^2+z+1)}{z^4}\,dz$$

where C is the LOWER ~~half~~ quarter circle centered at 0 joining $\frac{-1-i}{\sqrt{2}}$ and $\frac{1-i}{\sqrt{2}}$ in the positive (counterclockwise) sense.



$$I = \int_C \frac{z^3+z^2+z+1}{z^4}\,dz = \int_C \frac{dz}{z} + \int_C \frac{dz}{z^2} + \int_C \frac{dz}{z^3} + \int_C \frac{dz}{z^4}$$

$$\frac{-1-i}{\sqrt{2}} = e^{-i\frac{3\pi}{4}}, \quad \frac{1-i}{\sqrt{2}} = e^{-i\pi/4}$$

- $$\int_C \frac{dz}{z} = \int_C d\,\mathrm{Ln}\,z = \mathrm{Ln}\left(\frac{1-i}{\sqrt{2}}\right) - \mathrm{Ln}\left(\frac{-1-i}{\sqrt{2}}\right) = -i\frac{\pi}{4} - \left(-i\frac{3\pi}{4}\right) = i\frac{\pi}{2}$$

- $$\int_C \frac{dz}{z^2} = -\frac{1}{z}\Bigg|_{\frac{-1-i}{\sqrt{2}} = e^{-i3\pi/4}}^{\frac{1-i}{\sqrt{2}} = e^{-i\pi/4}} = -e^{i\pi/4} + e^{i3\pi/4} = -\frac{1+i}{\sqrt{2}} + \frac{-1+i}{\sqrt{2}} = -\sqrt{2}$$

- $$\int_C \frac{dz}{z^3} = -\frac{1}{2}\frac{1}{z^2}\Bigg|_{\frac{-1-i}{\sqrt{2}} = e^{-i3\pi/4}}^{\frac{1-i}{\sqrt{2}} = e^{-i\pi/4}} = -\frac{1}{2}\left(e^{i\pi/2} - e^{i\frac{3\pi}{2}}\right) = -\frac{1}{2}(i+i) = -i$$

- $$\int_C \frac{dz}{z^4} = -\frac{1}{3}\frac{1}{z^3}\Bigg|_{e^{-i3\pi/4}}^{e^{-i\pi/4}} = -\frac{1}{3}\left(e^{i3\pi/4} - e^{i\frac{9\pi}{4}}\right) = -\frac{1}{3}e^{i\frac{3\pi}{2}}\left(e^{-i\frac{3\pi}{4}} - e^{i3\pi/4}\right)$$
$$= \frac{i}{3}(-2i)\sin\frac{3\pi}{4} = \frac{2}{3}\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{3}$$

So: $$I = i\frac{\pi}{2} - \sqrt{2} - i + \frac{\sqrt{2}}{3} = -\frac{2\sqrt{2}}{3} + i\left(\frac{\pi}{2} - 1\right)$$

V. Let

$$f(z) = \frac{1}{(2-z)(z+3)}.$$

1. (2 pts) Write $f(z)$ as a sum of fractions, i.e.,

$$f(z) = \frac{A}{z-2} + \frac{B}{z+3};$$

$$A = \lim_{z\to 2} \left[(z-2)f(z)\right] = -\lim_{z\to 2} \frac{1}{z+3} = -1/5$$

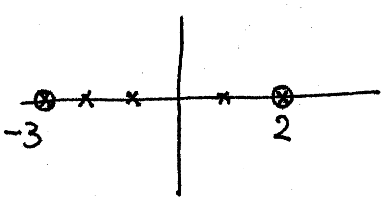
$$B = \lim_{z\to 3} \left[(z+3)\, f(z)\right] = \lim_{z\to -3} \frac{1}{2-z} = 1/5$$

$$f(z) = \frac{-1}{5} \frac{1}{z-2} + \frac{1}{5} \frac{1}{z+3}$$

$f(z)$ has singular points at $z = 2, -3$.

2. (3 pts) Explain whether it is possible to expand f(z) in Laurent (or Taylor) power series of:

(i) z, that converges in $0 \le |z| < 3$?



$f(z)$ "blows up" at $z = 2, -3$: it is NOT analytic there

The region $0 \le |z| < 3$ encloses $z = 2$.

So, we can NOT expand $f(z)$ in Laurent series in this region.

(ii) z, that converges in $3 < |z|$?

$f(z)$ is free of singular points in this region; so, $f(z)$ is analytic for $|z|>3$ and we CAN expand it in Laurent series there.

(iii) $z+1$, that converges in $1 < |z+1| < 4$?

The region $1<|z+1|<4$ encloses the points $z=2,-3$.

So, $f(z)$ is NOT analytic in $1<|z+1|<4$,

and therefore canNOT be expanded in Laurent series.

3. (4 pts) Write the Laurent series expansion of $f(z)$ for $\underline{5 < |z-2| < \infty}$ as a power series of $(z-2)$.

$$f(z) = \frac{1}{(2-z)(z+3)}.$$

Let $w = z-2$: $f(z) = \dfrac{1}{-w\,(w+5)}$, where $|w|>5$.

$$f(z) = -\frac{1}{w^2} \cdot \frac{1}{1+\underbrace{\frac{5}{w}}_{\lambda\,:\,|\lambda|<1}} = -\frac{1}{w^2}\left[1 - \frac{5}{w} + \left(\frac{5}{w}\right)^2 + \dots + (-1)^n\left(\frac{5}{w}\right)^n + \dots\right]$$

$$= -\frac{1}{w^2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{5}{w}\right)^n = -\sum_{n=0}^{\infty}(-1)^n\frac{5^n}{(z-2)^{n+2}}$$

$$\therefore \quad f(z) = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{5^n}{(z-2)^{n+2}}$$

VI. (6 pts) Let

$$f(z) = \frac{1}{(z^2+z)(z+2)^3}.$$

Compute the integral of $f(z)$ on the circles of <u>center 1</u> and radii 1/2, 3/2, and 100, respectively.

$f(z)$ has singular points at $z=0,-1,-2$.

<u>Radii</u>: • $R=\frac{1}{2}$; the circle C encloses <u>none</u> of the singular points.

So, for $R=\frac{1}{2}$, $\oint_C dz\, f(z) = 0$, by the Cauchy integral theorem.

• $R=\frac{3}{2}$; the circle C encloses the singular point $z=0$ only.

$$\oint_C dz\, f(z) = \oint_C dz \frac{\overbrace{\frac{1}{(z+1)(z+2)^3}}^{g(z)}}{z} = 2\pi i \cdot g(0),$$ by the Cauchy integral formula.

$$\Rightarrow \oint_C dz\, f(z) = 2\pi i \cdot \frac{1}{(0+1)(0+2)^3} = \frac{\pi i}{4}.$$

• $R=100$; the circle C encloses all singular points. Because $f(z)$ is analytic for $|z|>R$, C can be deformed with $R\to\infty$, <u>without</u> change in the result of integration.

$$\oint_C dz\, f(z) = \oint_{C(R\to\infty)} \frac{dz}{z^2\cdot z^3} = 0,$$ because $z^2+z \approx z^2$, $z+2 \approx z$ and $\oint_C z^n dz = 0$ for $n \neq -1$ where C encloses 0.

VII. (3 pts) Determine where in the complex plane the following functions are analytic ($\bar{z}$ is the complex conjugate of z):

(i) $\frac{e^z}{\sin z}$

This function is analytic at all z where $\sin z \neq 0$

$\Rightarrow$ $z \neq n\pi$, n: integer.

(ii) $z(\bar{z}+i)$

This function depends explicitly on $\bar{z}$, which is NOT analytic anywhere. So, the function is NOT analytic anywhere.

(iii) $e^{\frac{1}{z-1}}$

Let $w = \frac{1}{z-1}$

$$e^{\frac{1}{1-z}} = e^w = 1 + w + \cdots + \frac{w^n}{n!} + \cdots \quad : \text{converges}$$

for all $w \neq \infty \Leftrightarrow z \neq 1$

So, the function is analytic for $z \neq 1$.

VIII. (3 pts-BONUS) Determine the constant A so that the following function is analytic everywhere.

$$f(z) = \begin{cases} A\,\frac{\cosh z - 1}{z^2} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0. \end{cases}$$

For $z \neq 0$, $f(z) = A \dfrac{\cosh z - 1}{z^2}$;

so $f(z)$ is a ratio of two analytic functions and it is also analytic itself.

For $z \to 0$, $\quad f(z) = A \dfrac{\left(1 + \frac{z^2}{2} + \dots\right) - 1}{z^2} = \dfrac{A}{2}$.

So, we must have $\quad \dfrac{A}{2} = f(0) = 1 \iff A = 2$

For this value of A, $f(z)$ has a limit as $z \to 0$ that agrees with its value at $z=0$. So, $f(z)$ has a Taylor series at $z=0$, and this series converges for all z (as it would for $\cosh z$).

It follows that, for $A=2$, $f(z)$ is analytic everywhere.