DAVID SHIROKOFF: Hi, everyone. Welcome back. So today I'd like to take a look at gain and phase lag. And we're going to consider this simple problem.

So first off, find a periodic solution to x dot dot plus 8x dot plus 7x equals F_0 times cosine omega*t. So basically, it's just forcing a differential equation with some frequency cosine omega t. And the problem we're interested in today is to give the gain and phase lag to this periodic solution.

So I'll let you take a look at this problem and I'll be back in a minute.

Hi, everyone. Welcome back.

OK, so we're interested in finding this periodic solution to the differential equation. And we see here that we're forcing it with right-hand side of cosine omega*t. So the standard procedure is to first complexify. So we're going to consider the differential equation x dot dot plus 8x dot plus 7x equals F_0 e to the i*omega*t. And then at the end, we're going to take the real part of our solution to this differential equation. And we do this because cosine omega*t is the real part of e to the i omega t.

Now, for this equation, we see that we're forcing it with an exponential. So we can just use the exponential response formula. And this gives us a particular solution. So the exponential response formula is 1 over p of i*omega times the right-hand side, which is F_0 e to the i*omega*t.

And in this case, the characteristic polynomial, p of s, is s squared plus 8s plus 7. So that p of i*omega is going to be 7 minus omega squared-- so the omega squared comes from s squared-- plus 8i*omega.

Now, I'd just like to take a moment and step back for a second. If we take a look at the differential equation, the input on the right-hand side is F_0 e to the i*omega*t.

Now notice how this characteristic-- sorry, the exponential response formula gives us a particular solution, which is 1 over p of i*omega F_0 e to the i*omega*t, which is a periodic solution. But moreover, the output of this formula shows that it's actually the input forcing multiplied by this factor 1 over p of i*omega. So this factor right here, 1 over p of i omega relates the input forcing to the output forcing. Or sorry, to the output response. And this factor is sometimes called the complex gain.

And the reason it's the complex gain is because if we take a look at p of i*omega, it has a real part and an imaginary part. So specifically, p of i*omega contains two pieces of information. Because it's a complex number, we can think of it as being an amplitude and a phase. So the amplitude of p of i*omega is It's sometimes called the gain. And the phase of p of i*omega is the phase lag. So p of i*omega contains two pieces of information, which relate the input forcing signal to the output response of the differential equation.

Now, I'd like to go ahead and write x of t, decomplexify. So we're going to take the real part of 1 over p of i*omega times F_0 e to the i*omega*t. So we have 7 minus omega squared plus 8i*omega. Upstairs is F_0 e to the i*omega*t. And I'm going to just put this in amplitude-phase form.

And when I take the real part, I'll do it in two steps first. 8 omega squared. Square rooted. F_0 e to i*omega*t divided by e to the i*phi. Where here phi is the phase lag and tangent phi is going to be the imaginary part over the real part. And just quickly note that as omega goes from 0 to infinity, tangent phi will go from 0 to pi. So this is the range of phi that we're interested in.

And now, when we take the real part, we end up with F_0 1 over 7 minus omega squared quantity squared plus 8 omega squared, all squared rooted, cosine omega*t minus phi.

And now we'd like to look at what the amplitude and phase are as a function of omega. So for each fixed omega, the output is going to be a sinusoid which oscillates at the same frequency as the input. However, the only difference is that it's going to have a rescaled amplitude. And it will have a shifted phase as well.

So for the amplitude response, we're interested in plotting 1 over the amplitude of p of i*omega. So here's the amplitude.

And this function is just going to decrease and asymptotically approach infinity. So this is the amplitude response. This is 1 over 7 minus omega squared square rooted plus 8omega squared, quantity square rooted. This is omega.

And then in addition to the amplitude response, we also have the phase. And the phase, if I go back up here for a second, we can write it explicitly as phi is equal to tan inverse 8omega divided by 7 minus omega squared. So I'm going to plot omega on this axis and phi on this axis.

And, OK, so to plot this curve, we note that when omega is 0, tan inverse of 0 is 0. So we're going to start at 0.

Typically, what I usually like to do in this case is look at the denominator in the arctangent. We see that when omega is equal to the square root of 7, this argument blows up. It goes to infinity. And the arctangent of infinity is pi over 2. So I can draw a curve in here, which is pi over 2. So we know that phi is pi over 2 when omega is equal to root 7. And note how this is the natural frequency if there was no damping in the system.

And then lastly, there's another curve here at pi. We see that as omega approaches infinity, again, this argument approaches 0.

And if you were to take the derivative of this whole quantity, we would see that it's-- of tan inverse of this argument, we would see that the function's increasing for all omega. So it actually looks something like this. It looks like an s-shaped curve that asymptotically approaches pi as omega goes to infinity.

So just to quickly recap. When we look at the differential equation, or a particular solution to a differential equation which is being forced by a periodic sinusoidal function, the output is always going to be rescaled and phase shifted, but still oscillating at the same frequency. And specifically, depending on which frequency the differential equation's forced at, the amplitude will take on different values. The amplitude response will take on different values. And the phase shift will also have different values depending on the frequency of forcing.

So I'd just like to conclude here, and I'll see you next time.