18.03SC Practice Problems 4

Linear models

Solution Suggestions

1. *A tank holds V liters of salt water. Suppose that a saline solution with concentration of c grams/liter is added at the rate of r liters/minute. A mixer keeps the salt essentially uniformly distributed in the tank. A pipe lets solution out of the tank at the same rate of r liters/minute. Write down the differential equation for the amount (not the concentration!) of salt in the tank. Use the notation $x(t)$ for the number of grams of salt in the tank at time t. Check the units in your equation. Write it in standard linear form.*

The concentration of salt at any given time is $x(t)/V$ grams/liter, so for small Δt, we lose $rx(t)\Delta t/V$ gm from the output pipe, and we gain $rc\Delta t$ gm from the input pipe. This gives the differential equation $x'(t) = rc - \frac{rx(t)}{V}$. A check of units shows that both sides are in grams/minute. In standard linear form, this equation is

$$x' + \frac{r}{V}x = rc.$$

2. *Now assume that c and r are constant. In fact, suppose that $r = 2$ liters/minute and the volume of the tank is $V = 1$ liter. Solve this equation under the assumption that $x(0) = 0$.*

What is the limiting amount of salt in the tank? Does your result make sense by simple logic? When will the tank contain half that amount?

Under these assumptions the equation simplifies to $x' + 2x = 2c$. Solve by separation of variables to obtain $x = Ce^{-2t} + c$ for any constant C. At the given initial value $x(0) = 0$, $Ce^0 + c = 0$, so $C = -c$. Thus the solution that satisfies the given initial condition is
$$x(t) = -ce^{-2t} + c = c(1 - e^{-2t}).$$

The limiting amount of salt is $c = c \cdot 1 = cV$ grams since e^{-2t} tends to zero as $t \to \infty$. This is what we should expect, since the concentration in the tank should approach that of the inflow.

The tank will contain half of this limiting amount of salt when $c(1 - e^{-2t}) = c/2$, that is, when $t = \frac{\ln 2}{2}$.

3. *Now suppose that the outflow from this tank leads into another tank, also of volume 1, and that at time $t = 1$ the water in this second tank contains no salt. Again there is a mixer and an outflow. Write down a differential equation for the amount of salt in the second tank, as a function of time.*

Denote the amount of salt in the second tank at moment t by $y(t)$. The equation for the second tank is the same as that for the first tank, except with concentration of the inflow replaced by $x(t)$. That is, the differential equation for $y(t)$ is

$$y' + ry = rx(t).$$

4. *Draw a picture of a circuit with a voltage source, a resistor, and a capacitor, in series. Denote by $I(t)$ the current in the circuit, where the positive direction is, say, clockwise,*

and by $V(t)$ the voltage increase across the voltage source, at time t. Denote by R the resistance of the resistor and C the capacitance of the capacitor (in units which we will not specify)—both positive numbers. Then

$$R\frac{dI}{dt} + \frac{1}{C}I = \frac{dV}{dt}$$

Suppose that V is constant, $V(t) = V_0$. Solve for $I(t)$, with initial condition $I(0)$.

It is common to write the solution in the form $ce^{-t/\tau}$. Calculate c and τ. Note that τ is measured in the same units as t (in order for the exponent to be dimensionless). It is called the characteristic time for the system. What is $I(t + \tau)$ in terms of $I(t)$?

power source   capacitor
resistor
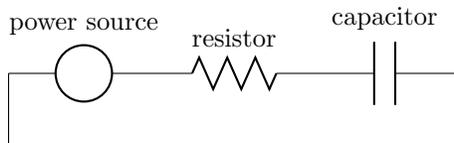
When V is constant, the equation becomes $R\frac{dI}{dt} + \frac{1}{C}I = 0$, which is separable. Solve to obtain

$$I(t) = I(0)e^{-\frac{t}{RC}}.$$

This can be written in the form $I(t) = ce^{-t/\tau}$ where $c = I(0)$ is the initial condition and $\tau = RC$ is the characteristic time. Then $I(t + \tau) = ce^{-(t+\tau)/\tau} = ce^{-\frac{t}{\tau}}e^{-1} = I(t)/e$.

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