First Order Response to Exponential Input

We start with an example of a linear constant coefficient ODE with exponential input signal.

Example 1. Solve $\dot{x} + 2x = 4e^{3t}$.

Solution. We could use our integrating factor, but instead let's use the method of *optimism*, i.e., the inspired guess. The inspiration here is based on the fact that differentiation reproduces exponentials:

$$\frac{d}{dt}e^{rt} = re^{rt}.$$

Since the right hand side is an exponential, maybe the output signal $x(t)$ will be also. Let's try

$$x_p(t) = Ae^{3t}.$$

This is not going to be the general solution, so we use the subscript p to indicate it is just one *particular solution*. We don't know what A is yet, but we will be led to its value by substitution. Substituting x_p into the DE we get

Left hand side: $\dot{x}_p + 2x_p = 3Ae^{3t} + 2Ae^{3t} = 5Ae^{3t}$.

Right hand side: $4e^{3t}$.

Equating the two sides we get

$$5Ae^{3t} = 4e^{3t} \Rightarrow 5A = 4 \Rightarrow A = 4/5.$$

So, we were led to the value of A and we have that one solution to the DE is

$$x_p(t) = \frac{4}{5}e^{3t}.$$

The associated homogeneous equation $\dot{x} + 2x = 0$ has general solution $x_h(t) = Ce^{-2t}$. By the superposition principle, we add x_p and x_h to get the general solution to our DE:

$$x(t) = x_p(t) + x_h(t) = \frac{4}{5}e^{3t} + Ce^{-2t}.$$