18.03 Problem Set 5, Second Half

The first half of this problem set was handed out on March 12 and is available on the web.

I encourage collaboration on homework in this course. However, if you do your homework in a group, be sure it works to your advantage rather than against you. Good grades for homework you have not thought through will translate to poor grades on exams. **You must turn in your own writeups of all problems, and, if you do collaborate, you must write on the front of your solution sheet the names of the students you worked with.**

Because the solutions will be available immediately after the problem sets are due, **no extensions will be possible**.

L17	F 12 Mar	LTI systems, superposition, review
L18	M 15 Mar	Engineering examples: an interview with Prof. Kim Vandiver: Damping ratio, review
R12	T 16 Mar	Exam review
L19	W 17 Mar	**Hour Exam II**
		III. Fourier series, Dirac delta function, and Laplace transform
R13	Th 18 Mar	Getting ready for Fourier series
L20	F 19 Mar	Fourier Series: EP 8.1
L21	M 29 Mar	Operations on Fourier Series: EP 8.2, 8.3
R14	T 30 Mar	Ditto
L22	W 31 Mar	Periodic solutions, resonance: EP 8.3, 8.4
R15	Th 1 Apr	Ditto
L23	F 2 Apr	Step function and delta function: SN 17

Part I. 20. (F 19 Mar) Notes 7A-1, 7A-2a.

21. (M 29 Mar) (a) Notes 7A-2a again, by writing the function in terms of sq(t).
(b) Notes 7A-2b, by integrating the Fourier series of the derivative.

Part I.17 answer. (correction) $x = (t^2 - 2) - \frac{1}{3}\cos(2t - 1)$.

Part II. 20. (F 19 Mar) [Fourier Series] This problem will use the Mathlet `Fourier Coefficients` at `http://math.mit.edu/mathlets`. When the applet opens you are presented with a series of sliders labeled b_n. By pressing the `[Formula]` radio button you can see that they are coefficients of sines in a Fourier series made up entirely of sine functions. If you press the `[Cosine]` radio button you'll see a_n's, coefficients of cosines. Move one of the slider handles: a cosine or sine curve appears and changes amplitude. Release it at some value and move another one. The white curves shows the new sinusoid, and the yellow curve shows the sum of the two. By moving more sliders you can build up more complicated sums and more complicated functions.

Now select the `Target [B]`. Is it an even function or an odd function? Based on this, decide whether to appoximate it using sines or cosines. Select one or the other appropriately (using `[All terms]`) and do the best you can by eyeballing the result to get the best approximation you can to the green target curve. Does it appear that only even terms are needed? Only odd term? or both? The `[Odd terms]` and `[Even terms]` buttons allow you to choose just the even terms or the odd terms, and gives you more of the one you select. If it seems that just the even or odd terms will be useful, explain (in words) why. (Supplementary Notes §16.1 may be helpful to you here.)

(a) Write down these values of the coefficients.

(b) The target function `[B]` is the periodic function with period 2π which is given by $f(t) = \frac{\pi}{4}$ for $-\frac{\pi}{2} < t < \frac{\pi}{2}$, $f(t) = -\frac{\pi}{4}$ for $\frac{\pi}{2} < t < \frac{3\pi}{2}$, and $f(\pm\frac{\pi}{2}) = 0$. Compute the Fourier coefficients for this function, using the integral formulas for them, and compare with your answers from **(a)**. Reset the sliders to the computed values and see if it looks like a better fit.

(c) Now set the sliders to some random set of values. Still with target function `[B]` displayed, select the `[Distance]` button. A number appears at the upper right corner of the screen. This is the "root mean square" distance from the target function to the selected finite Fourier sum. You can read about this distance in Supplementary Notes 16.4, but perhaps it is enough to know that it is a measure of goodness of fit. Instead of eyeballing the fit as before, start from the bottom and successively adjust the sliders to minimize the distance. Write down the optimal values of the coefficients. Compare with the computed values.

Lessons: (1) The Fourier coefficients are the coefficients resulting in the best possible fit, and **(2)** the process of optimizing one coefficient is independent of the process of optimizing any of the others. (This is "orthogonality.")

21. (M 29 Mar) [Fourier Series] **(a)** Find the Fourier series for $2\sin(t - \frac{\pi}{3})$ (Hint: A function of period 2π has just one expression as a linear combination of $\cos(mt)$'s and $\sin(nt)$'s.)

The square wave $\mathrm{sq}(t)$ is the odd function of period 2π such that $\mathrm{sq}(t) = 1$ for $0 < t < \pi$ and $\mathrm{sq}(\pi) = 0$. In class we calculated its Fourier series.

(b) $\mathrm{sq}(t)$ has minimal period 2π, but it is also a function of period 4π. Use the integral expressions (EP 8.2 (6)–(8)) for the Fourier coefficients to calculate its Fourier series, regarded as a function of period 4π. Comment on the relationship between your answer and the Fourier series for $\mathrm{sq}(t)$.

Use the Fourier series for $\mathrm{sq}(t)$, along with calculus and algebraic manipulations, to compute the Fourier series of each of the following functions without evaluating any of the integrals for the Fourier coefficients. In each case, sketch a graph of the function, as well, and give the minimal period.

(c) $\mathrm{sq}(t - \frac{\pi}{4})$. **(d)** $1 + 2\,\mathrm{sq}(2\pi t)$. **(e)** The $f(t)$ of `[B]` in the Mathlet.
(f) The periodic function $g(t)$ with period 2π such that $g(t) = t$ for $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$ and $g(t) = \pi - t$ for $\frac{\pi}{2} \le t \le \frac{3\pi}{2}$. (Hint: what is $g'(t)$ in terms of $f(t)$?)