18.02 Problem Set 3, Part II Solutions

1. (b) To get to the point P, start at the origin, add the vector $\langle 0, 2\rangle$ to go up to the center of the disk at $t = 0$; then add the vector $\langle t, 0\rangle$ to get to the center of the disk at time t; and finally (do what it takes to) shift over by one unit at an angle $\theta = t/2$, measured clockwise from the vertical (see below). How to do that last move? Just add the vector $-\langle \sin(t/2), \cos(t/2)\rangle$, as can be seen by from a sketch.
Combining, we get $\overrightarrow{OP}(t) = \langle 0, 2\rangle + \langle t, 0\rangle + \langle -\sin(t/2), -\cos(t/2)\rangle$ or

$$\mathbf{r} = \mathbf{r}(t) = \langle t - \sin(t/2),\ 2 - \cos(t/2)\rangle.$$

The reason that the angle is $\theta = t/2$ after time t is as follows: the center of the disk has moved t units to the right. Since the disk rolls without slipping, the horizontal distance it travels is equal to $r\,\theta =$ the circular arclength "unrolled" by the disk, where r is the radius and θ the central angle in radians. Since $r = 2$ and the horizontal distance is t, we get $t = 2\theta$.

(c) See applet. The shape of of the curve is an elongated periodic 'wave' created by the up-and-down movement of the point as the wheel moves to the right.

2. (a) $\mathbf{r}(t)$ is clearly in the plane through O defined by $\mathbf{u}$ and $\mathbf{v}$. But $|\ \mathbf{r}(t)\ |^2 = (\mathbf{u}\cos(t) + \mathbf{v}\sin(t)) \cdot (\mathbf{u}\cos(t) + \mathbf{v}\sin(t)) = \cos^2(t) + \sin^2(t) = 1$ since $\mathbf{u}\cdot\mathbf{u} = \mathbf{v}\cdot\mathbf{v} = 1$ and $\mathbf{u}\cdot\mathbf{v} = 0$ by assumption. So $\mathbf{r}(t)$ sweeps out the unit circle centered at O in $\mathcal{P}$.

b) We need to find two orthogonal unit vectors $\mathbf{u}$ and $\mathbf{v}$ which lie in the plane given by $x + 2y + z = 0$; then we can use the result of part(a). To get the first one, we can use take any vector orthogonal to $\mathbf{n} = \langle 1, 2, 1\rangle$, i.e. any vector $\mathbf{u}_1 = \langle p, q, r\rangle$ with $p + 2q + r = 0$, and then make it a unit vector. We'll pick $\langle 1, -1, 1\rangle$ to get $\mathbf{u} = \frac{1}{\sqrt{3}}\langle 1, -1, 1\rangle$. To get $\mathbf{v}$, we can just take the cross- product of $\mathbf{u}_1 \times \mathbf{n}$, and then also make it a unit vector. This gives $\mathbf{v}_1 = \langle 1, -1, 1\rangle \times \langle 1, 2, 1\rangle = 3\langle -1, 0, 1\rangle$ and then $\mathbf{v} = \frac{1}{\sqrt{2}}\langle -1, 0, 1\rangle$. With these choices we get

$$\mathbf{r}(t) = \frac{1}{\sqrt{3}}\langle 1, -1, 1\rangle \cos(t) + \frac{1}{\sqrt{2}}\langle -1, 0, 1\rangle \sin(t),$$

or $x = \dfrac{1}{\sqrt{3}}\cos(t) - \dfrac{1}{\sqrt{2}}\sin(t)$, $y = -\dfrac{1}{\sqrt{3}}\cos(t)$, $z = \dfrac{1}{\sqrt{3}}\cos(t) + \dfrac{1}{\sqrt{2}}\sin(t)$.

Note: other choices for $\mathbf{u}_1$ will give other vectors $\mathbf{v}$ and the resulting equations will look different, but they will be equivalent (i.e. give the same curve C).

3. a) Any such line Ł must have direction vector $\langle a, b, c\rangle$ which is normal to $\langle 1, 2, 1\rangle$. So we must have

$$a + 2b + c = 0.$$

Since the line Ł passes though O, we can write the equation of Ł in the point-direction form: $x = a\,t,\ y = b\,t,\ z = c\,t,$ and then use the equation $a + 2b + c = 0$ above, we get

$$x = a\,t, \quad y = b\,t, \quad z = (-a - 2b)\ t.$$

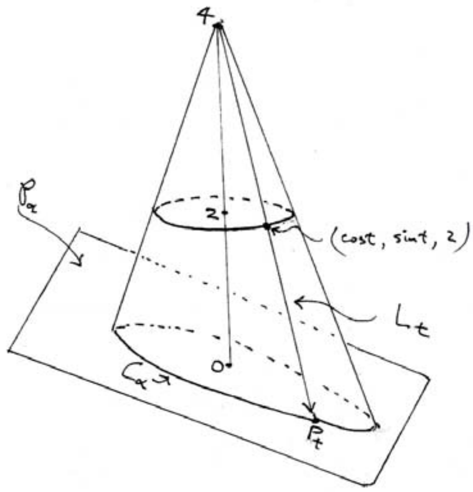
But Ł is described redundantly: scaling a and b equally gives an equivalent equation. To remove this redundancy, we need to look at two cases: either $b = 0$ in which case we get the line $x = t,\ y = 0,\ z = -t,$ or $b \neq 0$ in which case we may divide through by b to get

$$x = \alpha\,t, \quad y = t, \quad z = (-\alpha - 2)\ t$$

where the parameter α corresponds to a/b.

b) Since these lines all pass through O, they can be described by their unit direction vectors. The fact that the lines all lie in $\mathcal{P}$ means that their direction vectors all lie in $\mathcal{P}$. The tips of these unit vectors (considered as origin vectors) form a circle. So we have a circle's worth of lines through O lying in $\mathcal{P}$.
Note that the family of lines described algebraically in part(a) is a one-parameter family, with α as the parameter, and that this is consistent with the geometric fact that we have 1-dimensional circle of lines.

4. (a) Let P_t be the shadow of the point $(\cos t, \sin t, 2)$ on the plane $\mathcal{P}_\alpha$: $my + z = 0$ for an arbitrary but temporarily fixed value of t. See figure.



(b) Let P_t be the intersection of the line L_t joining (0,0,4) to $(\cos t, \sin t, 2)$ with the plane $\mathcal{P}_\alpha$. Let u be the parameter for L_t, then the vector-parametric equation for L_t is given by

$$L_t : \vec{R}_t(u) = \langle 0,0,4\rangle + u\langle \cos t, \sin t, -2\rangle$$

(since direction vector for L_t is $\langle \cos t, \sin t, 2\rangle - \langle 0,0,4\rangle$).

$P_t = L_t \cap \mathcal{P}_\alpha$: substitute $x = u\cos t, y = u\sin t, z = 4-2u$ into $my+z=0 \to mu\sin t + (4-2u) = 0 \to u^* = \frac{4}{2-m\sin t} \to P_t = \left(\frac{4\cos t}{2-m\sin t}, \frac{4\sin t}{2-m\sin t}, \frac{-4m\sin t}{2-m\sin t}\right)$ (after simplification). Thus $\vec{r}_\alpha(t) = \vec{OP}_t =$ the answer given (with $m = \tan\alpha$).

(c) $\alpha = 0$, so $\vec{r}_0(t) = \langle 2\cos t, 2\sin t, 0\rangle$ is indeed an enlarged circle in the x-y plane.

$|\vec{r}_\alpha(t) - \vec{r}_0(t)| = 2\sqrt{5}m|\frac{\sin t}{2-m\sin t}|$ (for $m>0$, after simplification). That gives the following differences:

$t=0$	$t=\pi/2$	$t=\pi$	$t=3\pi/2$
0	$\frac{2\sqrt{5}m}{2-m}$	0	$\frac{2\sqrt{5}m}{2+m}$

So the largest value is $\frac{2\sqrt{5}m}{2-m}$.

(The following paragraph is extra) To put this in perspective, if $\alpha = .1$ rad (so $m = .1003\ldots$), then this distortion is $0.236\ldots$ linear units. More generally, for small α, m is pretty close to α, so we can approximate the maximum distortion by $2\sqrt{5}\frac{\alpha/2}{1-\frac{\alpha}{2}} = \sqrt{5}\alpha(1 + \frac{\alpha}{2} + (\frac{\alpha}{2})^2 + \cdots)$ (geometric series), or $\sqrt{5}\alpha + O(\alpha^2)$.