Dot Product

The dot product is one way of combining ("multiplying") two vectors. The output is a scalar (a number). It is called the dot product because the symbol used is a dot. Because the dot product results in a scalar it, is also called the scalar product.

As with most things in 18.02, we have a geometric and algebraic view of dot product.

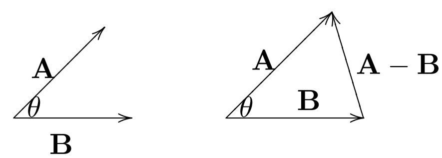
Algebraic definition (for 2D vectors):
If $\mathbf{A} = \langle a_1, a_2 \rangle$ and $\mathbf{B} = \langle b_1, b_2 \rangle$ then

$$\mathbf{A} \cdot \mathbf{B} = a_1 b_1 + a_2 b_2.$$

Example: $\langle 6, 5 \rangle \cdot \langle 1, 2 \rangle = 6 \cdot 1 + 5 \cdot 2 = 16.$

Geometric view:
The figure below shows $\mathbf{A}$, $\mathbf{B}$ with the angle θ between them. We get

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta$$



Showing the two views (algebraic and geometric) are the same requires the law of cosines

$$|\mathbf{A} - \mathbf{B}|^2 = |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2|\mathbf{A}||\mathbf{B}| \cos\theta$$
$\Rightarrow\ (a_1^2 + a_2^2) + (b_1^2 + b_2^2) - ((a_1 - b_1)^2 + (a_2 - b_2)^2) = 2|\mathbf{A}||\mathbf{B}| \cos\theta$
$\Rightarrow\ a_1 b_1 + a_2 b_2 = |\mathbf{A}||\mathbf{B}| \cos\theta.$
Since $\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2$, we have shown $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta$.

From the algebraic definition of dot product we easily get the the following algebraic law

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}.$$

Example: Find the dot product of $\mathbf{A}$ and $\mathbf{B}$.
i) $|\mathbf{A}| = 2$, $|\mathbf{B}| = 5$, $\theta = \pi/4$.
Answer: (draw the picture yourself) $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta = 10\sqrt{2}/2 = 5\sqrt{2}$.

ii) $\mathbf{A} = \mathbf{i} + 2\mathbf{j}$, $\mathbf{B} = 3\mathbf{i} + 4\mathbf{j}$.
Answer: $\mathbf{A} \cdot \mathbf{B} = 1 \cdot 3 + 2 \cdot 4 = 11$.

Three dimensional vectors
The dot product works the same in 3D as in 2D. If $\mathbf{A} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{B} = \langle b_1, b_2, b_3 \rangle$ then

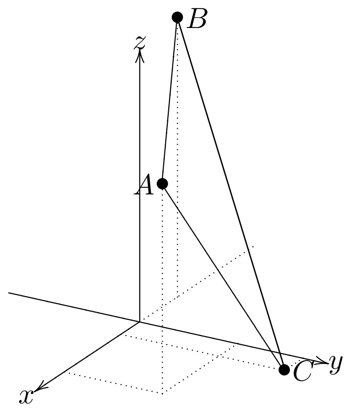
$$\mathbf{A} \cdot \mathbf{B} = a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3.$$

The geometric view is identical and the same proof shows

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta$$

Example:
Show $A = (4, 3, 6)$, $B = (-2, 0, 8)$, $C = (1, 5, 0)$
are the vertices of a right triangle.

Answer: Two legs of the triangle are $\overrightarrow{\mathbf{AC}} = \langle -3,\ 2,\ -6\rangle$ and $\overrightarrow{\mathbf{AB}} = \langle -6,\ -3,\ 2\rangle \Rightarrow \overrightarrow{\mathbf{AC}} \cdot \overrightarrow{\mathbf{AB}} = 18 - 6 - 12 = 0$. The geometric view of dot product implies the angle between the legs is $\pi/2$ (i.e $\cos\theta = 0$).



Definition of the term orthogonal and the test for orthogonality

When two vectors are perpendicular to each other we say they are *orthogonal*.

As seen in the example, since $\cos(\pi/2) = 0$, the dot product gives a test for orthogonality between vectors:

$$\mathbf{A} \perp \mathbf{B} \ \Leftrightarrow\ \mathbf{A} \cdot \mathbf{B} = 0.$$

Dot product and length

Both the algebraic and geometric formulas for dot product show it is intimately connected to length. In fact, they show for a vector $\mathbf{A}$

$$\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2.$$

Let's show this using both views.

Algebraically: suppose $\mathbf{A} = \langle a_1, a_2, a_3\rangle$ then

$$\mathbf{A} \cdot \mathbf{A} = \langle a_1, a_2, a_3\rangle \cdot \langle a_1, a_2, a_3\rangle = a_1^2 + a_2^2 + a_3^2 = |\mathbf{A}|^2.$$

Geometrically: the angle θ between $\mathbf{A}$ and itself is 0. Therefore,

$$\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}||\mathbf{A}|\cos\theta = |\mathbf{A}||\mathbf{A}| = |\mathbf{A}|^2.$$

As promised both views give the formula.