3.044 MATERIALS PROCESSING

LECTURE 2

Recap: Conduction Equation

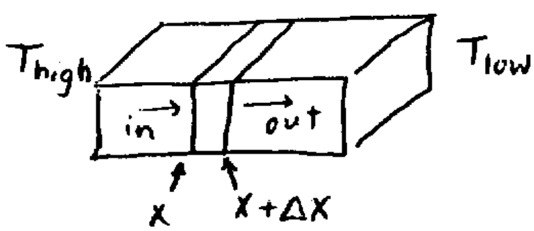
3D: $\rho\, c_p\, \frac{\partial T}{\partial t} = \nabla \cdot k\, \nabla T = k\, \nabla^2\, T$

1D: $\rho\, c_p\, \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\, k\, \frac{\partial T}{\partial x} = k\, \frac{\partial T^2}{\partial t^2}$

in our derivation last time we stated...

$$\begin{aligned}
\Delta x \frac{\partial H}{\partial T} &= q_{\text{in}} - q_{\text{out}} \\
&= q|_x - q|_{x+\Delta x} \\
&= \left(-k\frac{\partial T}{\partial x}\right)\bigg|_x + \left(k\frac{\partial T}{\partial x}\right)\bigg|_{x+\Delta x} \\
&= k\left(\frac{\partial T}{\partial x}\bigg|_{x+\Delta x} - \frac{\partial T}{\partial x}\bigg|_x\right) \\
&= k\frac{\Delta \frac{\partial T}{\partial x}}{\Delta x} \\
&= k\frac{\partial^2 T}{\partial t^2} \quad \text{assumes k independent of x, T}
\end{aligned}$$

Is it possible that the value of k is different at x and $x + \Delta x$?



Date: February 13th, 2012.

Taking k out of the derivative assumes that $k \neq f(x)$ and $k \neq f(T)$, because $T = f(x)$.

Is this assumption valid?

For <u>most</u> materials for <u>most</u> small working T ranges (< factor of 2) is usually negligible.

Simplify the conduction equation:

What we have done so far:

$$\rho\, c_p \frac{\partial T}{\partial t} = \nabla \cdot k\, \nabla T$$
$$\frac{\partial T}{\partial t} = \alpha\, \nabla^2\, T$$

3D to 1D

$$\frac{\partial T}{\partial t} = \alpha\, \frac{\partial^2 T}{\partial x^2}$$

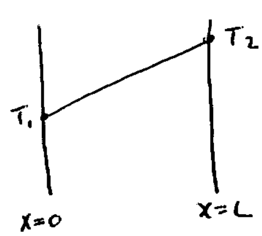
Assumption 1: Steady State

Steady State Conduction: unchanging temperature with time (T profile), heat is flowing, but at constant rates everywhere

$$\frac{\partial T}{\partial t} = \nabla^2 T = 0$$

$$\boxed{\nabla^2 T = 0} \quad \textbf{Laplace Equation}$$

1-D Sheet and Bar



$$\frac{\partial^2 T}{\partial x^2} = 0$$

Solve

$$\boxed{\frac{\partial^2 T}{\partial x^2} = 0}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) = 0$$

$$\partial\left(\frac{\partial T}{\partial x}\right) = 0$$

$$\frac{\partial T}{\partial x} = A$$

$$dT = A\,dx$$

$$\boxed{T = Ax + B}$$

Apply Boundary Conditions

$$\boxed{1.}\ @\ x = 0,\ T = T_1$$

$$T = A(0) + B = T_1$$

$$\therefore\ B = T_1$$

$$\boxed{2.}\ @\ x = L,\ T = T_2$$

$$T = A(L) + T_1 = T_2$$

$$\therefore\ A = \frac{T_2 - T_1}{L}$$

Plug In

$$T = \left(\frac{T_2 - T_2}{L}\right) x + (T_1)$$

Rearrange

$$\boxed{\frac{T - T_1}{T_2 - T_1} = \frac{x}{L}}$$

Define Dimensionless Variables:

LECTURE 2

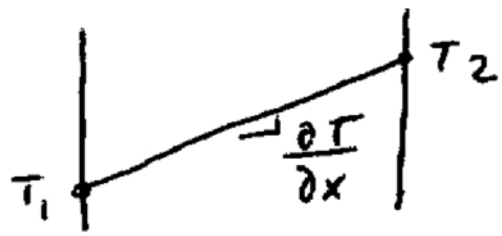
Dimensionless Position (0 - 1)

$$\Theta = \frac{\overbrace{T - T_1}^{\text{how far you are from } T_1}}{\underbrace{T_2 - T_1}_{\text{full temp. range}}} \quad \text{Fractional Position}$$

Dimensionless Position (0 - 1)

$$\chi = \frac{x}{L} \quad \text{Fractional Temperature}$$

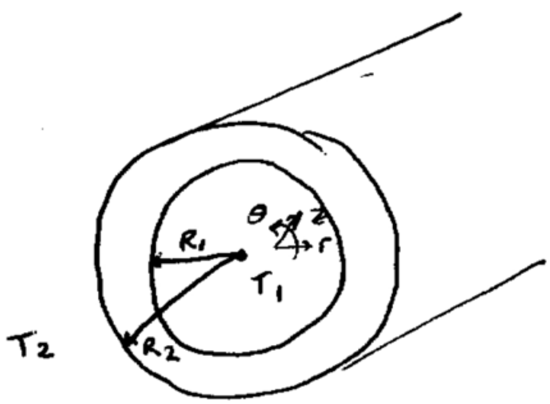
$$\boxed{\text{Solution: } \Theta = \chi}$$



$$q = - \overbrace{k}^{\text{constant}} \quad \underbrace{\frac{\partial T}{\partial x}}_{\text{slope = constant}}$$

$\therefore q$ is a constant

Heat flow out of a pipe



Steady State:

$$\nabla^2 T = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial Z^2} = 0$$

$$\frac{1}{}\frac{}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = 0$$

Solve

$$\frac{d}{dr}\left(r\frac{\partial T}{\partial r}\right) = 0$$
$$\int d\left(r\frac{\partial T}{\partial r}\right) = \int 0$$
$$r\frac{dT}{dr} = A$$
$$\int dT = \int \frac{A}{r}dr$$
$$\boxed{T = A\ln r + B}$$

Boundary Conditions

$$\boxed{1.}\ @\ r = R_1,\ T = T_1$$
$$T_1 = A\ \ln R_1 + B$$
$$\boxed{2.}\ @\ r = R_2,\ T = T_2$$
$$T_2 = A\ \ln R_2 + B$$

Solve for A

$$T_1 - A\ \ln R_1 = T_2 - A\ \ln R_2$$
$$T_1 - T_2 = A\ \ln R_1 - A\ \ln R_2$$
$$T_1 - T_2 = A\ \left(\ln\frac{R_1}{R_2}\right)$$
$$\boxed{A = \frac{T_1 - T_2}{\ln\frac{R_1}{R_2}}}$$

Solve for B

$$T_1 = A\ \ln R_1 + B$$
$$T_1 = \frac{T_1 - T_2}{\ln\frac{R_1}{R_2}}\ \ln R_1 + B$$
$$\boxed{B = T_1 - \frac{T_1 - T_2}{\ln\frac{R_1}{R_2}}\ \ln R_1}$$

Plug In

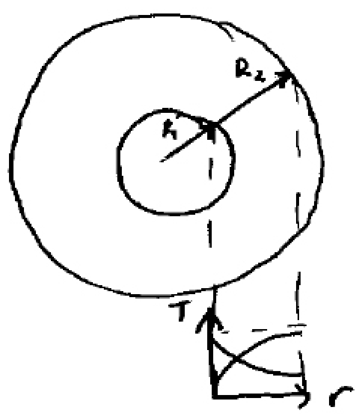
$$T = A \ln r + B$$

$$T = \frac{T_1 - T_2}{\ln \frac{R_1}{R_2}} \ln r + T_1 - \frac{T_1 - T_2}{\ln \frac{R_1}{R_2}} \ln R_1$$

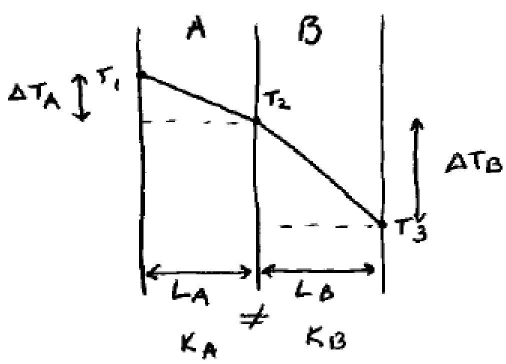
$$\boxed{\Theta = \frac{T - T_1}{T_2 - T_1} = \frac{\ln\left(\frac{r}{R_1}\right)}{\ln\left(\frac{R_2}{R_1}\right)}}$$

$$q = -k\frac{\partial T}{\partial r} \quad \text{Flux is not constant everywhere}$$

$$q \cdot \underbrace{A}_{2\pi r} = \text{constant} \quad \text{Total } \underline{\text{heat flow}} \text{ is constant everywhere}$$



Composite Wall



Steady State 1D

$$\frac{\partial^2 T}{\partial x^2} = 0 \quad \text{in material A and B}$$

Boundary Conditions

$$@\, x = L_A,\ T = T_2$$
$$@\, x = L_A,\ q_{\text{in}} = q_{\text{out}}$$

Solve

$$k_A \left.\frac{\partial T}{\partial x}\right|_{{L_A}^-} = k_B \left.\frac{\partial T}{\partial x}\right|_{{L_A}^+}$$
$$k_A \frac{\Delta T_A}{L_A} = k_B \frac{\Delta T_B}{L_B} \quad \text{because slope is const.}$$
$$\frac{k_A}{L_A}(T_1 - T_2) = \frac{k_B}{L_B}(T_2 - T_3)$$
$$\Rightarrow \text{Solve for } T_2\text{, the unknown } T$$

How is this useful to engineers?

$$\frac{\Delta T_A}{\Delta T_B} = \frac{\frac{L_A}{k_A}}{\frac{L_B}{k_B}}$$
$$\Delta T \propto \frac{L}{K}$$
$$\frac{L}{K} = \text{Thermal Resistivity}$$

Say we are making a furnace out of steel

$$\left.\frac{L}{k}\right|_{\text{steel}} = \frac{.01\text{m}}{30\frac{\text{W}}{\text{mK}}} = 0.0003 \qquad \Delta T \text{ 10x less}$$
$$\left.\frac{L}{k}\right|_{\text{mullite}} = \frac{.01\text{m}}{3\frac{\text{W}}{\text{mK}}} = 0.003 \qquad \Delta T \text{ 10x more}$$

Read As:

1. Mullite has 10x the temperature drop of steel
2. Mullite conducts slowly compared to steel
3. Steel is a <u>faster</u> conductor