Subject 24-242. Logic II. Answers to the Seventh Homework

1. Take a sentence α so that α is provably equivalent to $(Bew_{PA}([\ulcorner\alpha\urcorner]) \vee Bew_{PA}([\ulcorner{\sim}\alpha\urcorner]))$. Is α decidable in PA? Is it true?

   PA $\vdash (Bew_{PA}([\ulcorner\alpha\urcorner]) \rightarrow \alpha)$, and so, by Löb's Theorem, α is true and provable.

2. Show that, for each n, one can find an arithmetical formula τ_n such that, for each sentence ϕ, PA $\vdash([\ulcorner\phi\urcorner] < [n] \rightarrow (\tau_n([\ulcorner\phi\urcorner]) \leftrightarrow \phi))$.

   Let's say the sentences less that n are $\phi_1, \phi_2, \ldots, \phi_m$. We can take $\tau_n(x)$ to be the formula $((x = [\ulcorner\phi_1\urcorner] \wedge \phi_1) \vee (x = [\ulcorner\phi_2\urcorner \wedge \phi_2) \vee \ldots \vee (x = [\ulcorner\phi_m\urcorner] \wedge \phi_m))$.