Harvard-MIT Division of Health Sciences and Technology
HST.750: Modeling Issues in Speech and Hearing, Spring 2006
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Problem/Discussion Set for "Chaos and the Limits to Prediction"

1. **Sensitive Dependence on Initial Conditions.** Equations 25–27 in Lorenz (1963) define a system of coupled nonlinear equations known as the Lorenz equations.[1] They can be written in the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where $\mathbf{x}(\tau) = (X(\tau), Y(\tau), Z(\tau))$.

   (a) Using numerical integration (e.g., the function `ode23` in Matlab), integrate the Lorenz equations over the time interval $\tau \in [0, 100]$. Start your integration at the initial point $\mathbf{x}(0) = (20, 5, -5)$. Follow Lorenz and use the parameters values $\sigma = 10$, $b = 8/3$, $r = 28$, and a time increment $\Delta\tau = 0.01$. Plot the trajectory $\mathbf{x}(\tau)$ in three dimensions (e.g., using Matlab's `plot3`).[2]

   (b) Now integrate the Lorenz equations starting at 50 or more different nearby initial points chosen by selecting points $\mathbf{x}_i(0)$ at random from within a cube $\Delta X \Delta Y \Delta Z$ in phase space with dimensions $\Delta X = \Delta Y = \Delta Z = 10^{-3}$. Center your cube at the point $\mathbf{x} = (20, 5, -5)$. At each of the six times $\tau = \{6, 9, 12, 15, 18, 21\}$ plot the position of each of the 50 trajectories. Use a separate graph for each of the six time points, and, to facilitate comparisons, give all your plots the same scale. Use Lorenz's Eq. 37 to compute the approximate fractional change in volume of your cube that occurs over the total time interval $\tau = [0, 21]$. What happens to your cube of initial conditions?

2. **Lyapunov Exponent for the Tent Map.** The extent to which systems show sensitive dependence on initial conditions can be quantified by measuring (or calculating) what is known as the *Lyapunov exponent*. Consider two nearby initial conditions (or point) x_0 and $x_0 + \delta_0$, where the initial separation $|\delta_0|$ is small. How does the distance between the trajectories starting at these two initial points evolve under the operation of the map $x_{n+1} = f(x_n)$? To answer this question quantitatively, let the separation after n iterations be denoted δ_n. If the separation $|\delta_n|$ varies exponentially with n, then $|\delta_n| \simeq |\delta_0| e^{n\lambda}$, where λ is the Lyapunov exponent. Lyapunov exponents greater than zero ($\lambda > 0$) indicate that nearby trajectories diverge exponentially fast (i.e., chaotic behavior).

   (a) Show that
   $$\lambda \approx \frac{1}{n} \ln \left| (f^n)'(x_0) \right| \, .$$
   Hint: Solve for λ using the fact that $\delta_n = f^n(x_0 + \delta_0) - f^n(x_0)$ and take the limit $\delta_0 \to 0$. In these expressions, the superscripts on the function f represents functional composition (or iteration). For example, $f^2(x) \equiv f(f(x))$.

   (b) Use the chain rule to show that
   $$\lambda \approx \frac{1}{n} \sum_{i=0}^{n-1} \ln \left| f'(x_i) \right| \, .$$
   We now define the Lyapunov exponent by the limit $n \to \infty$:
   $$\lambda \equiv \lim_{n\to\infty} \left\{ \frac{1}{n} \sum_{i=0}^{n-1} \ln \left| f'(x_i) \right| \right\} \, .$$

   (c) One can generalize Lorenz's Eq. 35 by defining a "tent map" with the equations
   $$M_{n+1} = f(M_n) = \begin{cases} rM_n & \text{for } M_n < \frac{1}{2} \\ r(1 - M_n) & \text{for } M_n > \frac{1}{2} \end{cases}$$

[1] To see them in action, try the `lorenz` demo in Matlab.

[2] Extra credit: Compute the amount of computation time per time step required to solve the equations and compare with the value reported by Lorenz ("one second per iteration, aside from output time").

for $0 \le r \le 2$ and $0 \le M \le 1$. Plot $f(M_n)$, $f^2(M_n)$, and $f^3(M_n)$ assuming, as Lorenz did, that $r = 2$. Use your plots to verify the existence of "a single one-phase, a single two-phase, and two three-phase sequences, namely

$$\begin{array}{llll} 2/3, & \ldots, & & \\ 2/5, & 4/5, & \ldots, & \\ 2/7, & 4/7, & 6/7, & \ldots, \\ 2/9, & 4/9, & 8/9, & \ldots\text{"} \end{array}$$

(d) Calculate λ for the tent map. How does the long-term behavior of the system depend on the value of r?

3. **Orbit Diagram for the Quadratic Map.** Following Feigenbaum, define the iterated quadratic (or logistic) map as

$$x_{k+1} = 4\lambda x_k(1 - x_k)\ .$$

(a) Choose a random number x_0 in the interval $[0, 1]$. Plot the resulting sequence $\{x_k\}$ for $k = 0, 1, \ldots, 100$ for each of the following values of $r \equiv 4\lambda$: $r = \{2.8, 3.3, 3.5, 3.55, 3.9\}$. Repeat each for several different random choices for x_0.

(b) Now construct a plot that shows the long term behavior of x_k for many values of r simultaneously. To do this, first chose an initial value of r and some initial starting point, x_0. Then generate the resulting sequence $\{x_k\}$, iterating until the transients, which depend on the choice of x_0, have died down and the system has settled into its long term behavior (typically, iterating until $k = 300$ or so should be sufficient). Now generate many more points x_k (i.e., generate and store the values x_{301} to x_{600}) and plot the values of $x_{k>300}$ versus the value of r that you chose above. (In the xy-plane your plot will have points at 300 or so values of y—not necessarily all different—for every point, r, along the x-axis.) Now move to a nearby value of r and repeat the process. In this way, you'll build up a plot of $x_{k>300}$ versus r. Vary r over the range $r = [2.9, 4]$ with a resolution Δr of at least 0.002.

(c) Use your plot to estimate, as accurately as you can, the values r_n at which a cycle of period 2^n first appears (you may want to zoom in and compute certain regions of your plot with greater resolution). For example, you should find that a cycle of length 2 appears at $r_1 = 3$. Use your values of r_n to estimate the ratios

$$\delta_n = \frac{r_n - r_{n-1}}{r_{n+1} - r_n}$$

for $n = 2, 3$, and 4. Compare your answers with the limiting, universal value

$$\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.66920160910299067\ldots$$

known as Feigenbaum's constant.[3]

4. **Implications.**

(a) In an interview on National Public Radio, the physicist Peter Carruthers said that the phenomenon of chaos (i.e., sensitive dependence on initial conditions) "undermines the whole basis of science." What do you suppose he was talking about? Explain why you agree or disagree. If you disagree, what *does* chaos do to science?

(b) In the talk in which he coined the term "butterfly effect," Lorenz suggested that the flap of a butterfly's wings in Brazil can set off a tornado in Texas. Can it? Why or why not?

[3] In 1999, Feigenbaum's constant was computed to 1018 decimal places. (The record for π is now over 1.24 trillion decimal places.)