Harvard-MIT Division of Health Sciences and Technology
HST.750: Modeling Issues in Speech and Hearing, Spring 2006
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Problem/Discussion Set for "Dimensional Analysis" □

Problem 1. Systems of Units The units we choose to measure length, mass, and time are arbitrary human conventions and we are free to choose them for our convenience.

1. Some people use special units of mass, length, and time chosen so that the speed of light ($c = 2.998 \times 10^8$ m/s in MKS units), Planck's constant over 2π ($\hbar = 1.05457 \times 10^{-34}$ J s in MKS units), and Newton's gravitational constant ($G = 6.67 \times 10^{-11}$ m³/(kg s²) in MKS units) are all numerically equal to 1. What are the equivalent MKS values for these special units of length, mass, and time? When might you expect the use of these units to be convenient?

2. Some advanced SHBT students and faculty use special units of mass, length, and time chosen so that the speed of sound at room temperature ($c = 343$ m/s in MKS units), the Cambridge critical band at 1 kHz (CB = 132.64 Hz in MKS units, Glasberg and Moore, 1990), and the binaural minimum audible field at 1 kHz (MAF = 0 dB SPL in MKS units, ISO-389–7, 1996) are all numerically equal to 1. What are the equivalent MKS values for these special units of length, mass, and time? When might you expect the use of these units to be convenient?

Problem 2. A Simple Cochlear Model The simple transmission-line model of the cochlea discussed by Shera and Zweig (Sec. II–A of "A symmetry suppresses...") is defined by the variables/parameters

$$x, \ \omega, \ \bar{M}(x), \ M(x), \ R(x), \ C(x), \ \text{and} \ l \ .$$

What are the dimensions of each? How many independent dimensionless combinations (groups) can be formed? Identify and interpret the dimensionless groups used by Shera and Zweig.

Problem 3. Short Problems in Dimensional Analysis

1. *Radius of a black hole.* Use dimensional analysis to show that a black hole of mass M should have a radius of order GM/c^2, where G is Newton's gravitational constant and c is the speed of light. What is the approximate radius for a black hole with the mass of the sun ($M = 2 \cdot 10^{30}$ kg)?

2. *Deep-water waves.* Waves on the surface of a deep ocean travel with a speed v that depends on their wavelength λ, but not on their amplitude. Using dimensional analysis find how the wave speed should depend upon the wavelength and the density of the water.

3. *Vibration of a water drop.* When disturbed from their resting spherical shape, small liquid drops (e.g., raindrops or drops from a leaky faucet) can oscillate under the action of their own surface tension.[1] Using dimensional analysis, find how the period of oscillation depends upon the radius R and density ρ of the drop. Use your formula to estimate the period of vibration of a raindrop.

4. *Vibration of a star.* When disturbed from their resting spherical shape, large liquid drops (e.g., Lord Rayleigh's liquid-drop model of the sun) can oscillate under the action of their own gravity. Using dimensional analysis, find how the period of oscillation depends upon the radius R and density ρ of the drop. Use your formula to estimate the period of vibration of the sun.

5. *Roasting a turkey.* Cookbooks tell you to remove a turkey from the oven when its interior reaches a certain temperature (roughly 180°F in the thigh). Using dimensional analysis find how the total cooking time should depend on the size of the turkey. Assume that the turkey has characteristic dimension D, thermal conductivity k, mass density ρ, and specific heat at constant pressure C_p.[2] How do your results compare with the recommendations of the National Turkey Federation (see Table 1)?

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[1] Surface tension is defined as the force exerted by a surface per unit length over which it acts.

[2] The thermal conductivity of the turkey is defined as the rate of flow of heat through a unit area when a temperature gradient of one degree per unit distance is established perpendicular to that area. The specific heat of the turkey is defined as the quantity of heat required to raise a unit mass by one degree of temperature.

Approximate Roasting Times (hours)					
Weight (lbs)	8–12	12–14	14–18	18–20	20–24
Unstuffed	$2\frac{3}{4}$–3	3–$3\frac{3}{4}$	$3\frac{3}{4}$–$4\frac{1}{4}$	$4\frac{1}{4}$–$4\frac{1}{2}$	$4\frac{1}{2}$–5
Stuffed	3–$3\frac{1}{2}$	$3\frac{1}{2}$–4	4–$4\frac{1}{4}$	$4\frac{1}{4}$–$4\frac{3}{4}$	$4\frac{3}{4}$–$5\frac{1}{4}$

Table 1: Approximate roasting times at 325°F for a fresh or thawed turkey (National Turkey Federation, as quoted in the *Boston Globe*, 11/24/97).

6. *Mass in a tube.* A tube of mass M and length l is free to swing on a pivot. A mass m is positioned inside the (frictionless) tube at the end near the pivot. Initially horizontal, the tube is released and allowed to swing down under the influence of gravity. Let η be the fraction of the total length of the
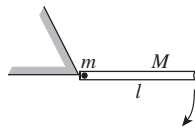
tube that the mass has traversed by the time the tube passes through the vertical. How does the fraction η depend on l?

7. *Ideal gas.* A box of volume V contains N particles of mass m, moving in various random directions with speed v. Using dimensional analysis find how the pressure of such a gas depends on N, V, m, and v.

8. *Planetary motion.* Show that Kepler's third law, $R^3/T^2 = $ constant, follows from the application of Newton's law of gravitation, $F = GMm/R^2$, to circular orbits. Use dimensional considerations.

Problem 4. First Digits of the Fundamental Physical Constants
The attached table from the National Institute of Standards and Technology (NIST) gives a "complete listing" of fundamental physical constants such as the speed of light, Planck's constant, the elementary charge, etc. For each value in the table for which a symbol is given, determine the first significant (i.e., non-zero) digit in its value. (For example, the first significant digit of c, the speed of light, is 2 in MKS units.) For simplicity, include values even when their symbols represent combinations of other constants on the list (e.g., e/h under "elementary charge").

1. Construct a histogram showing the number of times each of the nine digits 1–9 occurs in the first position.

2. You should find that roughly 1/3 of the numbers on the list begin with the digit 1. Can you think of a reason why the digit 1 is so prevalent?

3. The values in the table are generally given in MKS (SI) units. Would you have obtained a similar result if the values had been given in some other system of units (e.g., units based on the British inch and pound or the units used by advanced SHBT students)?

Problem 5. Physical Model of the Ear To help determine the vibratory pattern of the basilar membrane, Békésy built a physical model of the human cochlea in which the cochlear partition was represented by a thin rubber membrane. During construction he faced the following problem:

The width of the basilar membrane close to the stapes is about 0.1 mm and increases to about 0.5 mm at the apex of the cochlea. The fluid-filled canal above and below the basilar membrane is about 2 mm in diameter from one end of the cochlea to the other. It is difficult to carry out experiments with dimensions as small as these, and therefore it is desirable first of all to enlarge them in the model without altering any of the motional phenomena. If all the dimensions of the model are expressed in relative numbers based on one length l, which can be the length of the basilar membrane, then two such models with the same relative dimensions will themselves be alike. Also their motions will be similar, provided care is taken that, despite the change in length by some factor, the equation of motion remains the same.

Assume, as Békésy did, that "the movement of the membrane, as well as the surrounding fluid, is determined by a series of constants: the density ρ and viscosity η of the fluid, the frequency of movement of the stapes f, and the elasticity of the membrane ϵ."[3] Therefore, Békésy concluded, the equation describing the displacement, d, of the membrane at position x and time t resulting from a stapes velocity v_0 has the general form $\varphi(d, x, t, v_0, l, \rho, \eta, f, \epsilon) = 0$.

1. Use dimensional analysis to determine the necessary conditions for enlarging the cochlea while keeping its responses "geometrically similar" (i.e., identical up to a scale factor). Explain your reasoning.

2. Békésy chose to make his physical model roughly 5 times the size of the real cochlea. If the model is enlarged by this factor, by what factor must the volume elasticity of the rubber partition be changed to preserve similarity? Assume that Béksy filled his model cochlea with water (whose density and viscosity are not significantly different from those of the cochlear fluids).

3. If the model size and membrane elasticity are changed as calculated above, determine the approximate upper and lower frequency limits of the model basilar membrane (i.e., determine the maximum and minimum "characteristic frequencies" of the rubber partition). Assume that the normal human cochlea represents frequencies from 20 Hz–20 kHz.

Problem 6. Feeding Gulliver In Jonathan Swift's *Gulliver's Travels*, Gulliver describes how the Lilliputians determine his requirements for food and drink:

> "Having taken the height of my body by the help of a quadrant, and finding it to exceed theirs in the proportion of twelve to one, they concluded from the similarity of their bodies, that mine must contain at least 1728 of theirs, and consequently would require as much food as was necessary to support that number of Lilliputians."

1. Show that Swift's calculation is based on simple linear scaling in which limb diameters are proportional to length ($d \propto l$).

2. Deduce Galileo's scaling law from his famous diagram of bone scaling (Fig. 27). If a Lilliputian Galileo had applied this scaling formula, how many Lilliputian meals would he have found Gulliver to require?

3. If the Lilliputian Galileo had also known about Kleiber's metabolic scaling, how many Lilliputian meals would he have found Gulliver to require?

4. If a Lilliputian McMahon had done the calculation, how many Lilliputian meals would he have found Gulliver to require?

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[3]The volume elasticity, ϵ, is defined as the volume displacement per unit length of the membrane when a unit pressure difference is applied across it. For simplicity, Békésy ignored any difference in density between his rubber partition and that of the cochlear fluids.